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ON THE
DOCTRINE
OF THE
SPHERE.

BRITISH MUSEUM

ON THE
DOCTRINE
OF THE
SPHERE,
IN SIX BOOKS.

BOOK I. Containing ſome preliminary Properties of the CONE.
II. The General DOCTRINE of the SPHERE.
III. Of SPHERIC ANGLES and TRIANGLES.
IV. Of the ORTHOGRAPHIC PROJECTION.
V. Of the STEREOGRAPHIC PROJECTION.
VI. Of SPHERIC TRIGONOMETRY.

TO WHICH IS ADDED AN
APPENDIX:
CONTAINING THE
SOLUTION OF A PROBLEM,
FOR ASCERTAINING THE LATITUDE AND LONGITUDE OF A PLACE,
TOGETHER WITH THE APPARENT TIME.

By the REV. GEORGE WALKER, F. R. S.
—K

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BRITISH MUSEUM

TO

JOHN MANNING, M.D.

IN TESTIMONY OF
THAT ESTEEM AND AFFECTION
WHICH THE AUTHOR BEARS TO HIS VIRTUES;
BUT MORE PARTICULARLY
AS A TRIBUTE TO
THAT LONG AND CORDIAL FRIENDSHIP,
ON WHICH HAS BEEN FOUNDED
THE MOST PLEASING INTERCOURSE
OF THE AUTHOR'S LIFE,
THIS TREATISE IS
INSCRIBED

BY

GEORGE WALKER.

PREFACE.

THE Treatiſe which is here offered to the public, was compoſed nearly in its preſent form above twelve years ago, but with no other view than to my own amuſement in a ſcience to which I have perhaps been too much devoted, and to remove from my own mind the obſcurities, inaccuracies and inelegancies which diſgraced the doctrine of the ſphere in every treatiſe on the ſubject that I had ſeen. To accommodate the Students who attended my lectures at Warrington, while I had no immediate view of quitting its Mathematical chair, was the principal inducement to the publication. But this motive being more of a private than a public nature, will not vindicate me in ſending the work abroad: I can be juſtified only by the perſuaſion, that it will not be an unacceptable preſent to the lovers of pure Geometry, to whom it is reſpectfully preſented. The Elements of the Sphere, with the branches dependant thereon, have either been but imperfectly attempted in a geometric ſtyle, or been ſubjected to the ſlovenly hand of Algebra. The little Eſſay on Spherics of Mr. Simpſon of Glaſgow, annexed to his edition of the Elements of Euclid, is indeed perfect in its extent; but the general doctrine of the ſphere, which is the foundation of the whole; the geometry of ſpheric angles and triangles, together with the important ſcience of projection, are wholly neglected by him; and even the trigonometry of ſpheric triangles, which is his profeſſed object, is far from being complete, ſome of the moſt elegant and uſeful theorems being omitted. Had this maſterly genius thought fit to have extended his plan, the preſent treatiſe had aſſuredly not ſeen the light. I have not however borrowed from him or any other author, unleſs ſo far as the mind is neceſſarily poſſeſſed of the ideas which the converſation with the writings of others excites, and directly from Mr. Simpſon in the aliter of the XIII. Prop. B. III., and in my I. B. from Theodoſius. To what degree I have been aſſiſted by the latter author, I am unable now to remember. The I. Prop. B. VI. led me to the demonſtration of every theorem in ſpheric trigonometry, and with that eaſe and perſpicuity which without its aid I ſhould have deſpaired of, in propoſitions of ſo difficult a nature. It does indeed of itſelf at once reduce all ſpheric trigonometry to plane.

Though there ſhould be no other merit to recommend the preſent treatiſe to the Student, its utility will probably be acknowledged in the conſtruction of the Solid Figures, wherever they are required. If in this hope the author be miſtaken, his time has indeed been ill employed; for the firſt contrivance, and final preparation of theſe alone, has been much more fatiguing than all the other work beſides. To furniſh five hundred copies to the public, requires the cutting out above twenty thouſand figures; which are after all to be divided, pierced, fitted, and the whole inſerted in the plane to which they are adherent. If they ſhall be found to anſwer all the views of the author, the purchaſer will not wonder at the addition to the price of the work, which this moſt tedious and fatiguing buſineſs has occaſioned.

DIRECTIONS

FOR THE USE OF THE

SOLID FIGURES.

TABLE II.

Fig. 18. RAISE the plane ABC perpendicular to the table, the circular plane DEF being at the ſame time parallel to the table.

19. RAISE the triangular plane ABC perpendicular to the table, at the ſame time moving the circular planes DEF, IEL, till the points E in each coincide.

20. RAISE the circular planes EBDA, aBiH, of which the
21. former pierces the latter, till the points B, H in each coincide; at the ſame time turning the circles BQP, LWH, till they are perpendicular to the plane EBD.

22. RAISE the plane EHD, till it be perpendicular to the

table, and move the circle BQP on its axis, till it be parallel to the table.

Fig. 24. Raise the plane EPD perpendicular to the table, at the ſame time inclining the planes CPW, CPT towards EPD, till the points P in each coincide.

TABLE III.

25. Raise the planes DAC, DBC, till the letters D in each, or the parts of the ſingle letter F coincide.

26. n. 1. 2. Raise the plane ZPz perpendicular to the table; alſo
& 27. 28. move the plane ADB, which is pierced by ZPz, till the lines AB in each coincide. When the plane VBY is required, which is alſo pierced by ZPz, move it till the points B coincide.

29. Raise the plane EBP perpendicular to the table; and move the plane aBi, which is pierced by EBP, till the lines BH in each coincide. If the circle RSTV be required, which is alſo pierced by EBP, move it till it be perpendicular to the table.

30. 31. Raise the planes GB, ED, AF, CH perpendicular to the table.

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L E M M A T A.

L E M. I.

IF there be two triangles, which have two ſides of the one equal to two ſides of the other, each to each, and of the homologous angles, viz. ſubtended by equal ſides, if two be equal between themſelves, but of the other two, if each be greater, or each leſs than a right angle; then the triangles themſelves, the remaining homologous angles, and the remaining ſides, ſhall be equal between themſelves.

LET ABC, ADC be two triangles, having two ſides AB, AC of the one equal to two AD, AC of the other, each to *Fig.* 1. 2.

each, then may they be ſo placed as that two of the equal ſides ſhall coincide in one common ſide AC, and the triangles be ſituated towards oppoſite parts of AC. Of the homologous angles, viz. oppoſite to the equal ſides, let two ABC, ADC be equal between themſelves, but the remaining two ACB, ACD be either each greater, or each leſs than a right angle; I ſay the triangles themſelves, the angles ACB, ACD, and the remaining ſides BC, DC ſhall be equal alſo.

Because the angles ACB, ACD are each greater or leſs than a right angle, they are together greater or leſs than two right angles, wherefore BC, DC are not in one ſtrait line (13. E. 1.). Draw therefore the right line BD; becauſe AB is equal to AD, the angle ABD is equal to the angle ADB (5. E. 1.) but the angle ABC is, by hypotheſis, equal to the angle ADC, therefore the remaining angle CBD is equal to the remaining angle CDB, and conſequently CB is equal to CD (6. E. 1.). Wherefore the two ſides AB, BC being equal to the two AD, DC, each to each, and the angle ABC to the angle ADC, the triangles themſelves, and the remaining homologous angles ſhall be equal between themſelves (4. E. 1.).

LEM. II.

IF there be two triangles which have one angle of the one equal to one angle of the other, but in each this angle be either a right angle, or greater than a right angle, and of the ſides about the equal angles, if one ſide of the one be equal to one of the other, but the remaining ſide of the one greater than the remaining ſide of the other; then ſhall the baſe of the one be greater than the baſe of the other.

IN the triangles ABC, DEF let the angle ABC of the one be equal to DEF of the other, and each be either equal to or greater than a right angle; alſo of the ſides about theſe equal angles let AB be equal to DE, but BC greater than EF; I ſay the baſe AC ſhall be greater than the baſe DF. *Fig.* 3. 4. 5. 6.

FROM the greater BC take BG equal to the leſs EF. Becauſe AB is equal to DE, BG to EF, and the angle ABG to the angle DEF, the baſe AG is equal to the baſe DF (4. E. 1.). But becauſe the angle AGC is external to the angle ABC, it is greater than the angle ABC (16. E. 1.),

and

and ABC is either a right angle or greater than a right angle, conſequently AGC is greater than a right angle, and therefore greater than any other angle ACG of the triangle AGC (32. E. 1.). Wherefore the ſide AC ſubtending the greater angle, is greater than the ſide AG ſubtending the leſs (19. E. 1.). But AG has been ſhewn to be equal to DF, therefore AC is alſo greater than DF.

LEM. III.

IN equal circles a greater right line ſubtends a greater circumference, viz. comparing the circumferences which are not greater than a ſemicircle; but if the contrary, then the greater right line ſubtends the leſs circumference.

Fig. 7. 8. 9. Let ABG, DEI be two equal circles, in which are inſcribed the right lines AB, DE, ſubtending the circumferences AGB, DIE, of which neither is greater than a ſemicircle. If AB be greater than DE, I ſay the circumference AGB is greater than the circumference DIE.

7. 8. If the circumference AGB be a ſemicircle, then becauſe AB is greater than DE, and the circles are equal, DE is not a dia-

a diameter of the circle DIE, and therefore the circumferences DIE, DHG are unequal portions of the whole circumference of the circle, the one being leſs, the other greater than a ſemicircle. But DIE cannot be greater than a ſemicircle, wherefore it is leſs, viz. becauſe the circles are equal, it is leſs than the ſemicircle AGB.

But if the circumference AGB be greater than the circumference DIE, becauſe the whole circumferences of each circle are equal between themſelves, the remaining circumference ALB is leſs than the remaining circumference DHE.

If the circumference AGB be leſs than a ſemicircle, yet if AB be greater than DE; I ſay the circumference AGB is greater than the circumference DIE. To the centres C, F draw AC, BC, DF, EF. Becauſe the circles are equal, AC, CB are equal to DF, FE, each to each, but the baſe AB is greater than the baſe DE, wherefore the angle ACB is greater than the angle DFE (25. E. 1.). Draw CG making with AC the angle ACG equal to the angle DFE, and join AG. Becauſe the angle DFE, viz. the angle ACG is leſs than the angle ACB, the right line CG falls within the angle ACB; wherefore the point G is neceſſarily between the terms A, B, and the circumference AG is leſs than the circumference AGB. But in the equal circles, the equal angles ACG, DFE at the centres intercepting equal circum- *Fig.* 8. 9.

circumferences (26. E. 3.), the circumference AG is equal to the circumference DE. Wherefore the circumference AGB is alſo greater than the circumference DIE.

But if the circumferences ALB, DHE which are in each greater than a ſemicircle be compared, then is ALB leſs than DHE. For the whole circumferences of each being equal, but a part AGB of the one being greater than a part DIE of the other, the remainder ALB of the one is leſs than the remainder DHE of the other.

LEM. IV.

If in the ſame plane be drawn three or more right lines parallel to each other, right lines falling upon them, will be proportionally divided by them.

Fig. 10. 11. Let the right lines AC, BD, GE in the ſame plane be parallel between themſelves, and AB, CD be any two right lines falling upon them; I ſay AB, CD ſhall be proportionally divided in their concourſe with the parallels.

Let the right lines meet the parallels in A, B, E, and C, D, F. Join AD meeting EF in *G*. Becauſe the right line GFE is parallel to the baſes BD, AC of the triangles ABD, ADC, it will cut the ſides proportionally (2. E. 6.). Wherefore AB is to AE as AD to AG, and DC is to CF as AD to AG, therefore ex æquo, AB is to AE as DC to CF, viz. AB, CD are proportionally divided in E, F.

LEM. V.

IF three right lines parallel between themſelves or meeting in one common point fall upon two parallels, the parallels will be proportionally divided by the right lines falling upon them.

LET the three right lines EB, FC, GD parallel between themſelves, or having one common concourſe in A, fall upon the parallels EG, BD in E, F, G, and B, C, D; I ſay EG, BD ſhall be proportionally divided in F, C. *Fig.* 12. 13.

IF the three right lines be parallel between themſelves, then the right lines EG, BD falling upon the three parallels EB, FC, GD, are proportionally divided by them (LEM. 4.). 13.

BUT if the three right lines meet in one point A, then becauſe EG is parallel to BD, the triangles AEF, ABC, as alſo AFG, ACD are equiangular. Wherefore homologous ſides 12.

ſides are proportional (4. E. 6.), viz. EF is to BC as AF to AC, and alſo FG is to CD as AF to AC. Therefore ex æquo, EF is to BC as FG to CD, and by alternation, EF is to FG as BC to CD.

LEM. VI.

IF in a plane be two right lines parallel to each other, and from two points in each be drawn right lines perpendicular to any other plane, the right lines intercepted in the one plane between the perpendiculars ſhall be proportional to the right lines intercepted in the other.

Fig. 31. In the plane ABDC let the right lines AB, CD be parallel to each other, and from any two points A, B in the one, and C, D in the other let AE, BF, CG, DH be drawn perpendicular to a plane EFHG, alſo EF, GH be joined; I ſay AB, CD are proportional to EF, GH.

Because AE, BF, CG, DH are perpendicular to the ſame plane, they are parallel between themſelves (6. E. 11.), and therefore AEFB is one plane, as is alſo CGHD (7. E. 11.). But becauſe AB is alſo parallel to CD (Hyp.), the plane AEFB is parallel to the plane CGHD (15. E. 11.), and conſequently their common ſections EF, GH with the plane EGHD are parallel between themſelves (16. E. 11.). Draw AI, CL in the planes AF, CH parallel to EF, GH, and meeting BF, DH in I, L. Becauſe AI is parallel to EF, and CL to GH, but it has been ſhewn that EF is parallel to GH, AI will be parallel to CL (9. E. 11.). Wherefore the three ſtrait lines AB, BI, AI being parallel to the three CD, DL, CL, the triangles AIB, CLD, are equiangular (10. E. 11.), and AB is to CD as AI to CL (4. E. 6.). But AIFE, CLHG being parallelograms, the oppoſite ſides AI, EF, and CL, GH are equal between themſelves. Therefore alſo AB is to CD as EF to GH.

SCHOL. The demonſtration would have little differed, if AE, BF, CG, DH being mutually parallel each to each other, had been drawn in any inclination to the plane EFHG.

LEM. VII.

AN angle at the centre of a circle has the ſame proportion to four right angles, that the circumference on which it ſtands has to the whole circumference of the circle.

Fig. 14. Let ABD be a circle, C its centre, and ACB an angle at the centre ſtanding upon the circumference AB, the angle ACB ſhall be to four right angles, as the arch AB is to the whole circumference of the circle.

Produce AC to meet the circumference again in E, and through C draw the diameter DCF at right angles to AE. Then the ſemicircle ADE is biſected in D, and the ſemicircle AFE in F (30. E. 3.); wherefore the arch AD is a fourth part of the whole circumference of the circle. But the angle ACB is to the right angle ACD as the circumference AB is to the circumference AD (33. E. 6.), therefore quadrupling the conſequents, the angle ACB is to four right angles as the arch AB is to the whole circumference of the circle.

LEM. VIII.

If in two circles, two arches (one in each) ſubtend equal angles at the centres, as the one arch is to the whole circumference of which it is a part, ſo is the other arch to the whole circumference of which it is a part.

Let ABE, GIL be two circles, C, H their centres, and ACB, GHI two equal angles at the centres ſubtended by the arches AB, GI; the arch AB will be to the whole circumference ABE as the arch GI is to the whole circumference GIL. *Fig.* 14. 15. 16.

Because the arch AB is to the whole circumference ABE as the angle ACB is to four right angles (Lem. 7.), and for the ſame reaſon, the arch GI is to the whole circumference GIL as the angle GHI or the angle ACB is to four right angles. Therefore, ex æquo, the arch AB is to the whole circumference ABE as the arch GI is to the whole circumference GIL.

OF THE CONE.

BOOK I.

DEFINITIONS.

I. IF to the circumference of a circle from a point without the plane of the circle a right line be drawn, and the point remaining fixed through which the right line towards one extremity conſtantly paſſes; if towards the other extremity it be made to revolve in the circumference of the circle, till it return to the poſition from which it began to move; the ſuperficies deſcribed by the right line, ſituated on each ſide of the fixed point, and capable of being indefinitely extended (by the continuation of the deſcribing line towards both parts), are each of them called a CONICAL SUPERFICIES.

II. THE fixed point is called the VERTEX.

III. THE circle itſelf, the BASE.

IV. THE ſolid, comprehended under the baſe, and the conical ſuperficies, namely, intercepted between the vertex and baſe, is called a CONE.

V. THE AXIS, is the right line joining the vertex, and the centre of the baſe.

VI. RIGHT CONES, are thoſe whoſe axes are at right angles to their baſes.

VII. EVERY other, whoſe axis is not perpendicular to the cone, is denominated a SCALENE CONE.

VIII. IF a cone be cut by a plane, the figure bounded by its common ſection with the ſuperficies of the cone, is called a SECTION OF THE CONE.

PROP. I. THEOREM.

RIGHT lines drawn from the vertex to a point in the ſuperficies of the cone, are ſituated wholly in that ſuperficies.

FOR the right line deſcribing the conical ſuperficies paſſes through the vertex as a fixed point, and in its revolution muſt paſs through every point in the ſuperficies of the cone. It paſſes therefore through the given point, and is then a right line in the ſuperficies, and joining the ſaid point and the vertex. But only one right line can be drawn between two given points (10. Ax. E. 1.). Therefore a right line drawn from the vertex of a cone to a point in its ſuperficies, lies wholly in the ſuperficies.

COR. A right line drawn from the vertex of a cone to any point in the ſuperficies, falls, if produced, on the circumference of the baſe.

PROP. II. THEOREM.

IF a cone be cut by a plane through the vertex, the ſection will be a triangle.

For the plane cutting the cone, muſt paſs through ſome point in its ſuperficies; therefore the right line joining that point and the vertex will be both in the plane, and in the conic ſuperficies (Prop. 1.), and conſequently will be the common ſection of the plane, and the cone's ſuperficies, and if produced will fall upon the circumference of the baſe (Cor. Prop. I.). For the ſame reaſon is the oppoſite ſection of the plane with the cone's ſuperficies a right line, and falls upon the circumference of the baſe in ſome other point. Wherefore the plane cuts the baſe in two points; but it cuts it in a right line (3. E. 11.), viz. in the right line joining theſe two points. The ſection of the plane therefore with the cone is a figure bounded by three right lines, or it is a triangle.

Cor. The ſection of a cone through its axis, is a triangle, whoſe baſe is a diameter to the baſe of the cone.

DEF. IX.

If a ſcalene cone be cut by a plane through the axis at right angles to the baſe, making a triangular ſection (Prop. II.); and again by another plane perpendicular to the former, and intercepting from it towards the parts of the vertex a triangle ſimilar to the triangular ſection itſelf, but in a ſubcontrary poſition, (viz. ſo that the angles at the baſe of the intercepted triangle be equal to the alternate angles at the baſe of the triangular ſection) : this latter ſection of the cone is called a Subcontrary Section.

PROP. III. THEOREM.

If a cone be cut by a plane parallel to the baſe, the ſection will be a circle.

Let ABC be a cone, A the vertex, BGC the baſe; and let the cone be cut by a plane parallel to the baſe, whoſe ſection with the conic ſuperficies is the line DEF. I ſay DEF is a circle, having its centre in the axis of the cone. *Fig.* 18.

Let O be the centre of the baſe BGC, and join AO, which is the axis of the cone (V. Def.). Let AO, produced if neceſſary, meet the plane DEF in H, and through AO let a plane be paſſed, cutting the cone in the triangular ſection ABC (Prop. II.), the baſe in the diameter BOC (Cor. Prop. II.), and the plane DEF in the right line DHF. Becauſe the parallel planes BGC, DEF are cut by the plane ABC, their common ſections BC, DF are parallel (16. E. 11.). Wherefore the triangles AOB, AHD, and AOC, AHF are equiangular (29. E. 1.), and AO is to AH as BO is to DH; alſo as AO is to AH, ſo is OC to HF

(4. E. 6.). Therefore, ex æquo, as BO is to DH, ſo is OC to HF; but BO is equal to OC, therefore DH is equal to HF (14. E. 5.). To any other point E in the line DEF draw HE, and join AE. Becauſe AE is drawn from the vertex to a point E in the ſuperficies of the cone, it will meet the baſe (COR. PROP. I.). Let it meet it in G, and join OG. Becauſe the parallel planes BGC, DEF are cut by the plane AOG, the common ſections OG, HE will be parallel, and therefore AO will be to AH as OG is to HE. But it has been ſhewn that AO is to AH as BO to DH; wherefore ex æquo, OG is to HE as BO is to DH, and becauſe OG is equal to BO, HE will be equal to DH. By the ſame reaſoning is it ſhewn that right lines drawn from every point in the line DEF are equal to DH or HF, viz. that the line DEF is every where equidiſtant from a point H within it. Wherefore DEF is a circle, and H its centre.

COR. I. IF a cone be cut by a plane parallel to the baſe, and alſo by a plane through the axis, the common ſection of theſe planes is a diameter to the circular ſection made by the parallel plane.

COR. II. IF a cone be cut by a plane parallel to the baſe, the ſolid intercepted on the parts towards the vertex is itſelf a cone, having the ſame vertex, and the circular ſection made by the parallel plane for its baſe.

PROP. IV. THEOREM.

IF from any point in the circumference of a circle a perpendicular be drawn to a diameter thereof, the ſquare of the perpendicular ſhall be equal to the rectangle under the ſegments of the diameter. And Converſely, If there be a line paſſing through the extremities of a right line given in the ſame plane, and this line be of ſuch a property, that the ſquare of a perpendicular drawn from any point therein to the right line be equal to the rectangle under the ſegments of the right line, the line ſhall be the circumference of a circle, whoſe diameter is the right line given.

LET ABG be a circle, AD a diameter thereof, C its centre, and firſt, let a perpendicular from a point B therein drawn to AD fall in the centre C; I ſay, the ſquare of BC is equal to the rectangle ACD. For becauſe BC is equal to AC, the ſquare of BC is equal to the ſquare of AC; but AD being biſected in C, the ſquare of AC is equal to the rectangle ACD, therefore the ſquare of BC is equal to the rectangle ACD. *Fig.* 17.

AGAIN, Let a perpendicular drawn from a point E in the circumference meet AD in any other point F; I ſay, the ſquare of EF is equal to the rectangle AFD. Join CE, becauſe the angle EFC is right, the ſquares of EF, FC are equal (to the ſquare of EC (47. E. 1.), viz.) to the ſquare of AC. But becauſe AD is biſected in C, and unequally divided in F, the rectangle AFD together with the ſquare of FC is equal to the ſquare of AC (5. E. 2.). Wherefore the ſquares of EF, FC are equal to the rectangle AFD together with the ſquare of FC. Take away from each the common ſquare of FC, and the ſquare of EF is equal to the rectangle AFD.

CONVERSELY, If ABG be a line meeting in the ſame plane the right line AD in the points A, D, and the line ABG be of ſuch a property, that the ſquare of a perpendicular drawn from any point therein to AD be equal to the rectangle under the ſegments of AD; I ſay the line ABG ſhall be a circle, whoſe diameter is AD.

BISECT AD in C, and firſt, let a perpendicular drawn from a point B in the line ABG meet AD in C. Becauſe AD is biſected in C, the ſquare of AC is equal to the rectangle ACD, but the ſquare of BC is alſo equal to the rectangle ACD, therefore the ſquare of BC is equal to the ſquare of AC, and BC itſelf equal to AC.

AGAIN, Let a perpendicular drawn from any other point E in the line ABG meet AD in F, and join CE. Becauſe AD is biſected in C, and unequally divided in F, the rectangle AFD together with the ſquare of FC is equal to the ſquare of AC (5. E. 2.). But the rectangle AFD is equal to the ſquare of EF, therefore the ſquares of EF, FC are together equal to the ſquare of AC. But EFC being a right angle, the ſquares of EF, FC are together equal to the ſquare of EC (47. E. 1.). Wherefore the ſquare of EC is equal to the ſquare of AC, and EC itſelf equal to AC. By the ſame reaſoning is it ſhewn that the diſtance of every point in the line ABG from the point C is equal to AC or CD, and the points A, D are themſelves at the ſame diſtance from the point C; therefore the line ABG is a circle, and AD a diameter thereof.

PROP. V. THEOREM.

THE ſubcontrary ſection of a ſcalene cone is a circle.

LET ABG be a ſcalene cone, whoſe vertex is A, baſe BGC, and let ABC be the triangular ſection of the cone at right angles to the baſe; and IEL a ſubcontrary ſection at right angles to the plane ABC, and intercepting the triangle AIL *Fig.* 19.

AIL ſimilar to the triangle ABC, but in a ſubcontrary poſition (viz. ſo that the angles ALI, AIL are equal to the alternate angles ABC, ACB of the triangle ABC); I ſay, the ſection IEL is a circle.

FROM any point E in the line IEL draw EH perpendicular to the plane ABC, EH will fall on IL the common ſection of the planes (38. E. 11.); let it meet IL in H, through which in the plane ABC draw the right line DHF parallel to BC the baſe of the triangle ABC. Through any point G in the baſe draw GO perpendicular to ABC meeting BC in O. Becauſe the right lines GO, EH are perpendicular to the ſame plane, they are parallel between themſelves (6. E. 11.), and the right lines BOC, DHF are alſo parallel, therefore the plane of the baſe BGO is parallel to the plane paſſing through EH, DH (15. E. 11.), and conſequently the ſection DEF made by the plane DHE is a circle (PROP. III.), whoſe diameter is DF (I. COR. PROP. III.). But EH, being perpendicular to the plane ABC, is perpendicular to the right lines DHF, IHL (III. DEF. E. 11.), wherefore the rectangle DHF is equal to the ſquare of EH (PROP. IV.). Again, Becauſe the angle AIL is equal (to the angle ABC, viz.) to the angle ADF, and the oppoſite angles FHI, DHL are alſo equal, the triangles FHI, DHL are equiangular. Wherefore DH is to IH as HL is to HF (4. E. 6.), and the rectangle DHF is equal

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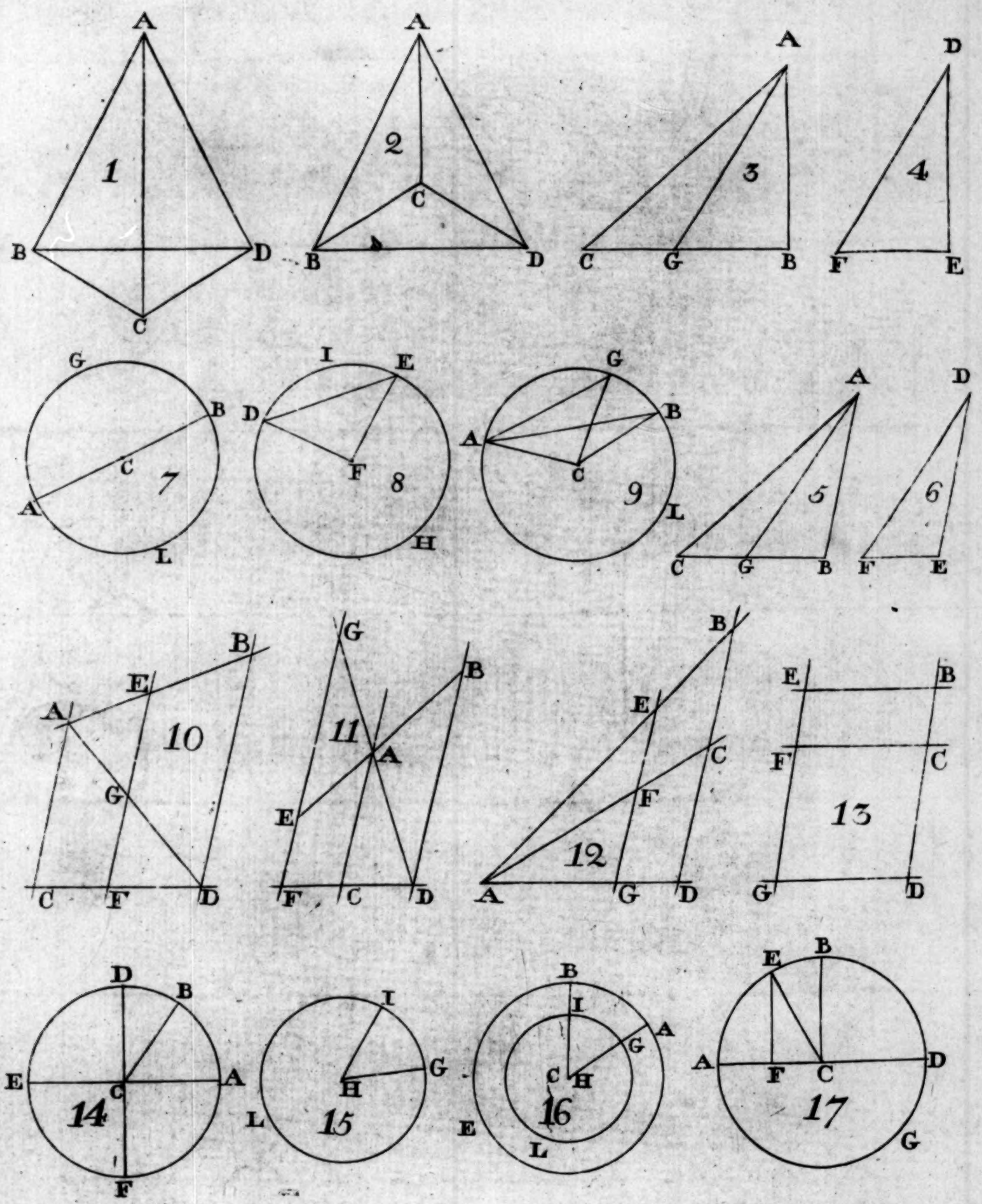
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TAB. 1. *pag. 22.*

to the rectangle IHL (16. E. 6.). But it has been ſhewn that the rectangle DHF is equal to the ſquare of EH, therefore the rectangle IHL is alſo equal to the ſquare of EH. In the ſame manner is it ſhewn that the ſquare of the perpendicular to IL drawn from whatever point of the line IEL is equal to the rectangle under the ſegments of IL. Wherefore the line IEL is a circle (PROP. IV.).

COR. The right line in which the triangular ſection through the axis at right angles to the baſe of the cone cuts the ſubcontrary ſection, is a diameter to the ſubcontrary ſection.

PROP. VI. THEOREM.

IF in the plane of a cone's baſe a right line be drawn touching the circumference, a plane paſſing through this touching line and the vertex of the cone, touches the cone in the right line joining the vertex and the point of contact in the circumference of the baſe.

LET the right line CK in the plane of the baſe BGC of the cone ABC touch the circumference in C, and join the points

Fig. 18.

points C and the vertex A; I ſay, the plane drawn through AC, CK touches the cone in the right line AC.

For the right line AC is in the ſuperficies of the cone (Prop. I.), and it is alſo in the plane ACK, therefore the plane ACK does meet the conic ſuperficies in the right line AC. But it meets it in no point which is not in the right line AC; for let P be a point in the ſuperficies, and not in AC. Join AP, which, if produced, meets the circumference of the baſe (Cor. Prop. I.); let it meet in Q, and join CQ. Becauſe CK touches the circumference in C, the point Q is not in the right line CK (Cor. II. E. III.), wherefore the planes ACQ, ACK, being not one and the ſame plane, meet each other in no point which is not in the right line AC their common ſection (I. and III. E. XI.). The point P is not therefore in the plane ACK, nor for the ſame reaſon is any other point of the ſuperficies which is not in AC. The plane ACK therefore touches the cone in the right line AC (Def. IX.).

OF THE SPHERE.

BOOK II.

DEFINITIONS.

I. A SPHERE is a ſolid body, generated by the revolution of a ſemicircle about its diameter as an immoveable axis.

II. The centre of the generating circle is alſo the Centre of the ſphere.

III. A right line drawn from the centre to any point on the ſuperficies of the ſphere, is called a Radius of the ſphere.

IV. A DIAMETER of the ſphere is any right line paſſing through the centre, and terminated on both ſides by the ſuperficies of the ſphere.

V. A plane is ſaid to touch the ſphere when meeting it in one point, it does not cut it, though produced on all ſides.

COR. 1. RIGHT lines drawn from the centre of the ſphere to every point in the ſuperficies, or all the radii, are equal between themſelves.

THIS is an immediate conſequence of DEF. 1.

COR. 2. HENCE all the diameters of the ſphere are equal between themſelves, and every diameter is biſected in the centre.

PROP. I. THEOREM.

IF a ſphere be cut by a plane, the figure bounded by the common ſection of the plane and the ſpheric ſuperficies, is a circle.

CASE 1. LET the ſphere be cut by a plane through the centre; I ſay the plane figure made by the ſection, and bounded by the ſuperficies, is a circle.

FOR becauſe the centre of the ſphere is in the plane figure made by the ſection, and the line bounding it is in the ſphere's ſuperficies, therefore right lines drawn from the centre to the line bounding the figure, are all radii of the ſphere. But the radii of the ſphere are equal between themſelves (1. COR. DEF.), therefore theſe right lines are all equal between themſelves, and the figure is a circle whoſe circumference is the line bounding the ſection on the ſuperficies of the ſphere.

CASE 2. LET the plane cutting the ſphere not paſs through the centre, viz. EBaD being a ſphere, C its centre, let BQP be the plane figure made by the ſection; I ſay BQP is a circle, whoſe circumference is the line BQP on the ſurface of the ſphere bounding the figure. Since the centre of the ſphere is not in the plane BQP, draw the right line CO from the centre at right angles to BQP (11. E. 11.), meeting the plane in O, and from O to any points B, Q, P in the line BQP draw OB, OQ, OP, alſo join CB, CQ, CP. Becauſe CO is perpendicular to the plane BQP, it is alſo perpendicular to all the right lines OB, OQ, OP drawn from the point O in the plane BQP (3. DEF. E. 11.) *Fig.* 20.

E. 11.); wherefore the triangles COB, COQ, COP are all right-angled at O, and conſequently the angles CBO, CQO, CPO are each leſs than a right angle. But becauſe CB, CQ, CP are equal between themſelves (1. COR. DEF.), and CO is common, each of the triangles have two ſides of the one equal to two of the other, and of the homologous angles, viz. ſubtended by theſe equal ſides, one in each is of equal magnitude, being a right angle, and the other is leſs than a right angle; therefore the remaining ſides in each triangle, viz. OB, OQ, OP are equal between themſelves (LEM. I.). Therefore the figure BQP is bounded by a line which is every where at an equal diſtance from a point O within it, viz. it is a circle, whoſe circumference is the line BQP bounding the ſection on the ſurface of the ſphere, and whoſe centre is O.

DEFINITIONS.

VI. A SECTION of a ſphere made by a plane cutting it through the centre, is called a GREAT CIRCLE.

VII. BUT if the plane do not cut the ſphere through the centre, the ſection is called a LESS CIRCLE.

VIII. THE POLES of any circle of the ſphere, are the extremities of a diameter of the ſphere at right angles to the circle.

IX. PARALLEL CIRCLES of the ſphere are thoſe whoſe planes are parallel.

X. A SPHERIC ANGLE is the inclination of the circumferences of two great circles of the ſphere, which meet each other, not being in the ſame plane.

XI. A SPHERIC TRIANGLE is a figure on the ſurface of the ſphere, comprehended by three arches of three great circles.

XII. Two circles of the ſphere are ſaid to TOUCH each other, whoſe circumferences meeting each other in one point, fall on both ſides of the concourſe without each other.

COR. Parallel circles of the ſphere have the ſame poles, becauſe the ſame diameter is perpendicular to them (CONVERSE. 14. E. 11.).

COR. 1. PROP. I. THE centre of the ſphere is the common centre of all great circles.

COR. 2. PROP. I. THE poles of a leſs circle, its centre, and the centre of the ſphere are in the ſame right line, viz. in a diameter of the ſphere at right angles to the centre; and therefore,

3. A RIGHT line paſſing through one of theſe four points at right angles to the plane of the leſs circle, will paſs through the other three, and a right line paſſing through two of theſe points, will paſs through the other two, and be perpendicular to the plane of the circle.

4. PARALLEL circles of the ſphere have their centres in the ſame diameter of the ſphere at right angles to each of them.

5. Two different great circles cannot be deſcribed through the ſame two points on the ſurface of the ſphere.

FOR only one and the ſame plane can be paſſed through the two given points, and the centre of the ſphere (2. E. 11.).

PROP.

PROP. II. THEOREM.

ALL great circles of the ſphere mutually biſect each other, and their common ſection is a diameter to the ſphere.

FOR the centre of the ſphere is the common centre of any two great circles (1. COR. PROP. I.), therefore the circles cutting each other, their common ſection is a diameter to the ſphere, and to both the circles, and conſequently the circles mutually biſect each other.

COR. A plane cutting the ſphere through the centre, biſects all the great circles of the ſphere (unleſs that which coincides with the plane itſelf), and conſequently biſects the ſphere itſelf.

PROP. III. THEOREM.

A GREAT circle perpendicular to the plane of another circle paſſes through the poles of that other circle, and their common ſection is a diameter to the other circle; Alſo, A great circle which paſſes through the poles of another, is perpendicular to that other, and cuts it through its centre.

FOR ſince the centre of the ſphere is in the plane of the great circle, a ſtrait line drawn through this centre at right angles to the other circle, will fall in the plane of the great circle (38. E. 11.), and both the poles and centre of the other circle are in this right line (8. DEF. and 2. COR. PROP. I.), therefore the great circle does paſs through the poles of the circle to which it is perpendicular, and paſſing through its centre, alſo, their common ſection is a diameter of the other circle.

AND converſely, Since the poles of a circle, its centre and the centre of the ſphere are in the ſame right line (DEF. 8. and 2. COR. PROP. I.), viz. in a diameter of the ſphere at right angles to the circle, therefore a great circle paſſing through the poles of another circle, does paſs through its centre, and alſo through a right line perpendicular to it, viz. the great circle is itſelf perpendicular thereto (18. E. 11.)

COR. 1. Two great circles perpendicular to each other, paſs mutually through the poles of each other.

COR. 2. IF a great circle paſs through the centre of a leſs, it will paſs through the poles alſo of the leſs, and therefore be perpendicular to the leſs.

PROP. IV. THEOREM.

THE pole of any circle of the ſphere is at an equal diſtance from every part of its circumference.

LET BQP be any circle of the ſphere, R, r its poles, from either of which draw RB, RQ to any points B, Q in its circumference; I ſay, RB, RQ are equal between themſelves. *Fig.* 20. 21.

JOIN Rr, meeting the circle in O, viz. in the centre of the circle (8. DEF. or 3. COR. PROP. I.), alſo join OB, OQ. Becauſe RO is alſo, for the ſame reaſon, perpendicular to the circle BQP, it will be perpendicular to OB, OQ (3. DEF. E. 11.), wherefore in the triangles ROB, ROQ ſince the angles at O are equal, being each right, the ſide RO common to both, and OB equal to OQ, the baſe RB will be equal to the baſe RQ (4. E. 1.)

By the ſame reaſoning are right lines drawn from either of the poles to any other points in the circumference ſhewn to be equal between themſelves, therefore the pole of every circle of the ſphere is equidiſtant from its circumference.

Cor. 1. Great circles paſſing through the pole of another circle, have equal arches intercepted between the circumference of the circle, and its pole.

For all great circles of a ſphere, being deſcribed round the ſame centre at an equal diſtance, are equal between themſelves, and the ſubtenſes of their intercepted arches are by this Prop. equal. Therefore, becauſe equal right lines in equal circles ſubtend equal arches (28. E. 3.), the arches themſelves are equal.

Cor. 2. The arch of a great circle paſſing through the pole of any other circle, and intercepted on both ſides of the pole by the circumference, is biſected in the pole.

Cor. 3. If a great circle paſs through the pole of another great circle, the arch intercepted from the pole to the circumference, is a fourth part of the great circle.

For great circles mutually biſecting each other (Prop. II.), the whole arch intercepted by the circumference of the other on both ſides of its pole is a ſemicircle, but this arch is biſected in the pole; therefore each part from the pole to to the circumference is a fourth part of the great circle, and equal to the fourth part of any other great circle of the ſame ſphere.

PROP. V. THEOREM.

IF through a point on the ſurface of the ſphere any number of right lines be drawn to fall on the circumference of a circle of the ſphere, thoſe two, which are drawn in the plane of a great circle paſſing through the pole of the former circle, are the greateſt and leaſt; viz. that is the greateſt which has the pole between the point from which it is drawn and its concourſe with the circle, the other is the leaſt; and that which is nearer to the greateſt is greater than that which is more remote.

LET I be any point on the ſurface, BQP any circle of the ſphere, and R the pole of BQP towards the ſame parts with the point I, from which point let any number of right lines *Fig.* 20. 21.

lines IB, IT, IQ, IP be drawn to fall on the circumference BQP, whereof IB, IP are drawn in the plane of a great circle EID paſſing through the points I, R; I ſay, IB which has the pole R ſituate between its extremes I, B is the greateſt of all the right lines ſo drawn, IP which has the pole without its extremes is the leaſt; and of the others, any one IT which is nearer to IB, is greater than IQ which is more remote.

THE great circle BIP is perpendicular to the circle BQP, and their common ſection BP is a diameter to BQP (PROP. III.). Draw RO, IS perpendicular to the plane BQP (11. E. 11.), and join ST, SQ. Then RO, IS will fall on the common ſection BP (38. E. 11.), and O will be the centre of the circle BQP (3. COR. PROP. I.); and becauſe RO, IS are parallel (6. E. 11.), but the point R is between the points B, I, the centre O will be between the points B, S. Wherefore SB is greater than ST, and ST than SQ, and SQ than SP (7. and 8. E. 3.). But IS, being perpendicular to the plane BQP, is perpendicular to all the right lines SB, ST, SQ, SP; therefore the triangles ISB, IST, ISQ, ISP have one angle at S in each a right one, and of the ſides about theſe right angles, one IS is common to them all, while the other is in each greater than that in the ſucceeding triangle, therefore the remaining ſide in each triangle is greater than the remaining ſide in the ſucceeding triangle (LEM. II.), viz. BI is greater than IT, and IT than IQ, and IQ than IP.

COR. IF from a point on the ſurface of the ſphere any number of great circles be drawn to fall on the circumference of a circle of the ſphere, then of the arches intercepted between the point and the circumference on which they fall, viz. taking thoſe circumferences in each, which are not greater than a ſemicircle; that is the greateſt which paſſes through the pole of the circle, but the remaining intercepted arch of the ſame great circle is the leaſt; and of the others, that which is nearer to the pole is greater than that which is more remote.

FOR all great circles are equal, and their common centre, viz. the centre of the ſphere, lies towards the ſame parts of the arches; but the right lines ſubtending them have been ſhewn to be in this order greater and leſs, therefore the arch ſubtended by a greater right line is greater than that ſubtended in an equal circle by a leſs (LEM. III.).

PROP. VI. THEOREM.

THE diſtance of the neareſt poles of two circles of the ſphere is the meaſure of their inclination to each other.

CASE 1. IF the circles be parallel, they have no inclination to each other, and their poles coinciding, the diſtance of their poles becomes nothing.

CASE 2. IF the circles be at right angles to each other, and be both great circles, they paſs mutually through the poles of each other, wherefore the poles of each being mutually in the circumferences of each other, the diſtance of their poles is the diſtance of a great circle from its own pole, viz. it is the fourth part of a great circle, or a quadrant (3. COR. PROP. IV.), which is the meaſure of a right angle.

CASE 3. WHEN the circles are inclined to each other in any leſs angle.

Fig. 20. 21. LET BaH, EaD be two great circles of the ſphere inclined to each other in any angle leſs than a right one, whoſe neareſt poles are P, I, and aCi a diameter of the ſphere their common ſection (PROP. II.). To C the centre of the ſphere draw CP, CI, and let the plane PCI cut the ſphere in the great circle EID (6. DEF.), and the inclined circles in the right lines BCH, ECD. Becauſe P is the pole of the circle BaH, the arch PB is a quadrant, as for the ſame reaſon is the arch IE. Theſe arches are therefore equal between themſelves, and taking away the common arch IB, the remaining arch PI is equal to the remaining arch BE. But becauſe the great circle EID paſſes through the poles of each of the circles BaH, EaD, it is perpendicular to each of them (PROP. III.), and therefore their common ſection aCi is perpendicular to the plane EID (19. E. 11.). Wherefore aC is perpendicular to CB, CE, and conſequently the angle BCE is the inclination of the planes BaH, EaD (6. DEF. E. 11.). But the arch BE is the meaſure of the angle BCE (VII. LEM.), and the arch BE has been ſhewn to be equal to the arch PI, wherefore the arch PI is the meaſure of the inclination of the great circles.

AND it is the ſame if one or both of the circles had been leſs ones, for they would have the ſame poles, and the ſame inclination to each other as the great circles which are parallel to them.

COR. 1. HENCE the inclination of two great circles is meaſured by the intercepted arch of a great circle paſſing through the poles of each of them, viz. of a great circle whoſe poles are the interſections of the circumferences of the two great circles.

FOR the great circles BaH, EaD being each perpendicular to the circle EBD, do each paſs through the poles of

EBD (PROP. III.), but the circumferences of the circles meeting in no other points than a, i, the extremities of their common ſection, a, i are the poles of EBD. And it has been ſhewn that the intercepted arch BE is the meaſure of the inclination of the circles to each other.

COR. 2. IF two circles be inclined to each other, and a great circle be deſcribed through their poles, viz. a plane be deſcribed at right angles to the plane of each of the inclined circles, the angle comprehended by the common ſections of this perpendicular plane with each of the inclined planes is the meaſure of the inclination.

COR. 3. IF BQP be a circle of the ſphere inclined to the circle EaD, and through R, I their neareſt poles, and the centre C of the ſphere a plane be paſſed cutting the ſphere in the great circle EPD, and the circles themſelves in the diameters BP, ED (PROP. III.); the arches BI, IP are the greateſt and leaſt diſtances of the pole I from the circle BQP (COR. PROP. V.), and the arches BI, IP together (if the pole I be without the terms B, P); but their exceſs (if within), is double to the inclination of the circles.

TOWARDS the parts of I oppoſite to B make the arch IN equal to the arch IB; becauſe the arch BN is double to the arch BI, and the arch BP double to the arch BR

(2. COR. PROP. V.), the whole or remainder PN is double to the whole or remainder IR. But PN is the ſum or difference of the arches BI, IP (accordingly as the pole I is without or within the terms B, P), and IR meaſures the inclination of the circles BQP, EaD to each other, therefore, &c.

PROP. VII. THEOREM.

A PLANE perpendicular to a diameter of the ſphere at either extremity of the diameter, touches the ſphere in that extremity, and no other plane can be paſſed through the point of contact, which does not cut the ſphere.

LET FCPK be a ſphere, C its centre, CP a diameter thereof, and WPT a plane perpendicular to CP at either extremity P; I ſay, the plane WPT touches the ſphere in the point P. *Fig.* 24.

LET V be any other point in the plane WPT, and join CV, PV. Becauſe the points P, V are in each of the planes WPT, CPV, the right line PV joining them is the common ſection of the planes. CP therefore, being perpendi-

cular

cular to the plane WPT, is perpendicular to PV, and conſequently in the triangle CPV, CV which is oppoſite to the right angle is greater than either ſide CP (32. and 19. E. 1.), viz. the point V is at a diſtance from the centre C of the ſphere greater than CP a ſemidiameter of the ſphere. But the ſuperficies of the ſphere is every where at a diſtance from the centre equal to CP (1. Cor. Def.), therefore the point V is without the ſphere, as for the ſame reaſon is every point in the plane WPT unleſs the point P only. The plane WPT therefore touches the ſphere in the point P (5. Def.).

I say further, that unleſs the plane WPT at right angles to CP, every other plane paſſing though P cuts the ſphere.

Let OPL be any other plane paſſing through P and not at right angles to CP. Draw CL perpendicular to the plane OPL, and join PL. Becauſe PL drawn in the plane OPL meets CL, it will meet it at right angles, wherefore in the triangle CPL, the right line CP which ſubtends the right angle CLP is greater than the ſide CL; therefore the point L is nearer to the centre than the point P in the ſuperficies, viz. the right line PL, and conſequently the plane OPL falls within the ſphere.

COR. ONLY one plane can touch the ſphere in the ſame point.

PROP. VIII. THEOREM.

A PLANE touching the ſphere, touches it only in one point.

FOR the plane does not cut, or fall within the ſphere (5. DEF.), if therefore it meet the ſphere in two points, theſe muſt be on its ſuperficies. But the right line joining theſe two points will be in the touching plane, and being alſo in the circumference of a great circle drawn through the two points, it will fall within that great circle (2. E. 3.), viz. within the ſphere itſelf. Wherefore the touching plane itſelf falls within the ſphere, which is contrary to the ſuppoſition.

PROP. IX. THEOREM.

IF a plane touch the ſphere, the ſemidiameter of the ſphere drawn to the point of contact will be perpendicular to the touching plane.

Fig. 24. LET FCPK be a ſphere, C its centre, and WPT a plane touching the ſphere in the point P; I ſay, CP, being joined, will be at right angles to the plane WPT.

IN the plane WPT draw any right line PV. Since the plane CPV paſſes through the centre of the ſphere, the ſection will be a great circle CPD, whoſe circumference is the line PD on the ſurface of the ſphere (PROP. I.). To any point V in the right line PV draw CV; becauſe the plane WPT touching the ſphere meets its ſuperficies only in the point of contact P (PROP. VIII.), the point V is without the ſphere, and therefore without the circle PD. For the ſame reaſon is every point in the right line PV, unleſs the point P without the circle PD, and therefore PV touches the circle in P (3. DEF. E. 3.), and conſequently meets CP at right angles (18. E. 3.). In the ſame manner is it ſhewn that CP is at right angles to every right line drawn through P in the plane WPT, and therefore it is at right angles to the plane itſelf (3. DEF. E. 11.).

COR. IF a plane touch the ſphere, every great circle paſſing through the point of contact, is perpendicular to the touching plane.

FOR the great circles, paſſing through the centre of the ſphere alſo, will paſs through the ſemidiameter drawn to the point of contact. But this ſemidiameter is at right angles to the touching plane, therefore every great circle, paſſing through this ſemidiameter, is at right angles to the touching plane (18. E. 11.).

PROP. X. THEOREM.

IF there be two planes, of which one touches, but the other cuts the ſphere through the point of contact, the common ſection of the planes will touch the circular ſection made by the cutting plane.

LET a plane WPT touch the ſphere in the point P, and GPF cut it through the ſame point, making the circular ſection GPF; I ſay, TP the common ſection of the planes will touch GPF in the point P. *Fig.* 24.

FOR the right line TP being in the touching plane meets the ſphere only in the point P (PROP. VIII.), and therefore the circumference of the circle in no other point. Therefore TP, being alſo in the plane of the circle GPF, touches its circumference in the point P (3. DEF. E. 3.).

PROP.

PROP. XI. THEOREM.

IF two circles of the ſphere touch each other, the common ſection of their planes will touch each of the circles.

Fig. 20. LET BaHi, BQP be two circles of the ſphere touching each other in the point B, and BY be the common ſection of their planes; I ſay, BY touches each of the circles in the point B.

FOR ſince the circumferences, and therefore the circles included by them, fall entirely without each other, unleſs in the point B (12. DEF.), every point which is common to the planes of both the circles, unleſs B only, falls without the circles. But the right line BY is common to both the planes, and therefore falls without both the circles, unleſs in the point B. Wherefore BY being drawn in the plane of each circle, and falling on the circumference of each in the point B only, touches each circle in the point B (2. DEF. E. 3.)

COR. 1. IF any number of circles of the ſphere touch each other in the ſame point, they will all have the ſame common ſection.

FOR only one right line can be drawn through the ſame common point in the plane of each to touch each circle.

COR. 2. IF the common ſection of the planes of two circles of the ſphere touch each circle, the circles themſelves will touch each other.

FOR ſince every point that is common to the planes of both the circles is found in their common ſection, but this common ſection falls without the circles unleſs in the common point of contact, the circles therefore meeting in this point only, fall on both ſides without each other, i. e. they touch each other (12. DEF.).

PROP. XII. THEOREM.

IF two circles of the ſphere touch each other, a great circle paſſing through the point of contact and poles of the one, will paſs through the poles of the other alſo, and therefore will be perpendicular to each of them.

LET BaHi, BQP two circles of the ſphere mutually touch each other in the point B; I ſay, a great circle EID paſſing *Fig.* 20.

paſſing through the point of contact B and the poles of either of them, as BaH, will paſs through the poles of BQP alſo.

BECAUSE EID paſſes through the poles of BaH, it is perpendicular thereto, and BH their common ſection is a diameter to BaH (PROP. III.). Wherefore BY the common ſection of the circles BaH, BQP being a tangent to each of them at the point B (PROP. XI.), will be at right angles to BH a diameter of one of them (18. E. 3.). But BH is the common ſection of the planes BaH, EID, and theſe planes are perpendicular to each other, conſequently BY which is drawn in one of them BaH at right angles to BH, will be at right angles to the other plane EID (4. DEF. E. 11.). Wherefore the plane EID will be perpendicular to every plane paſſing through BY (18. E. 11.), viz. it will be perpendicular to BQP, and being alſo a great circle, it will paſs through the poles of BQP (PROP. III.).

COR. IF two circles of the ſphere touch each other, the point of contact, their poles, and centres will be in the plane of one and the ſame great circle, which is perpendicular to each of the touching circles.

PROP.

PROP. XIII. THEOREM.

IF two circles of the ſphere meet each other, and a great circle perpendicular to each of them paſs through the point of their concourſe, the circles will touch each other in that point.

Let two circles BaH, BQP meet each other in the point B on the ſurface of the ſphere, and a great circle EID perpendicular to each of them paſs through the point B of their concourſe; I ſay, the circles will touch each other in B. *Fig.* 20.

Because the circles BaH, BQP are each perpendicular to EID, their common ſection BY will be perpendicular to EID (19. E. 11.), and therefore will be perpendicular to every right line drawn from the point B in the plane EID, viz. to BH, BP, which are the common ſections of EID with the circles BaH, BQP. But BH, BP are alſo diameters of the circles BaH, BQP (Prop. III.), therefore BY touches each circle in the point B (Cor. 16. E. 3.), and conſequently the circles themſelves touch each other (2. Cor. Prop. XI.).

Cor. Hence if two circles meet each other on the ſurface of the ſphere, and a great circle paſs through the poles of each and point of concourſe, the two circles will touch each other.

For the great circle paſſing through the poles of each, is perpendicular to each.

PROP. XIV. THEOREM.

TWO equal and parallel circles of the ſphere are equidiſtant from the centre of the ſphere; and if two parallel circles be equidiſtant from the centre of the ſphere, they will be equal.

Let BQP, LWH be two equal and parallel circles of the ſphere; I ſay, they are equidiſtant from C the centre of the ſphere.

Through the centre C draw a right line perpendicular to either of the parallel circles (11. E. 11.), and therefore to the other alſo (Converse. 14. E. 11.). Let this perpendicular meet the planes in O, U, which points of concourſe will be the centres of the circles (3. Cor. Prop. I.).

Let any plane paſſing through OU cut the circles in BP, LH, and join CB, CH. Becauſe CO, CU are perpendicular to the planes BQP, LWH, they are perpendicular to OB, UH; wherefore the triangles COB, CUH being right-angled at O, U, have the angles COB, CUH equal between themſelves, and the angles BCO, HCU each leſs than a right angle; alſo the homologous ſides, viz. CB, BO of the one are equal to CH, HU of the other, each to each; therefore the remaining ſides CO, CU are equal between themſelves (Lem. I.). But CO, CU are the diſtances of the circles from the centre C, therefore the circles are equidiſtant from the centre of the ſphere.

Conversely, If the parallel circles BQP, LWH be equidiſtant from the centre of the ſphere, the circles ſhall be equal.

For, the ſame things remaining, it is by the ſame reaſoning proved, that the angles COB, CUH are equal being right, the angles CBO, CHU each leſs than a right angle, and the ſides BC, CO equal to the ſides HC, CU, each to each; therefore the remaining ſides OB, UH are alſo equal (Lem. I.); viz. the circles themſelves, of which OB, UH are the ſemidiameters, are equal between themſelves.

Cor. 1. On the ſphere there can be only two equal and parallel circles, and theſe lie on oppoſite ſides of the centre of the ſphere.

For, if more than two could be equal and parallel, then two of them muſt lie on the ſame ſide of the ſphere's centre, but being equidiſtant from this centre, and alſo parallel, they would coincide together, and become one and the ſame circle, therefore, &c.

Cor. 2. If the circumferences of two equal and parallel circles be cut by a great circle at right angles to them both, the oppoſite points of ſection are diametrically oppoſite on the ſurface of the ſphere.

Thus the circumferences of the circles BQP, LWH are cut in the oppoſite points B, H, and P, L by the great circle drawn through OU. And becauſe the ſides and angles of the triangles BOC, HUC have been ſhewn to be equal, the angle BCO is equal to the angle HCU. But BC falling upon the right line OU makes the angles BCO, BCU together equal to two right angles, therefore the angles HCU, BCU are alſo equal to two right angles, and conſequently BC, HC are in one ſtraight line (14. E. 1.); viz. the points B, H are in a right line paſſing through the centre

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centre of the ſphere, as for the ſame reaſon are the points P, L.

PROP. XV. THEOREM.

If there be two equal and parallel circles of the ſphere, a great circle touching the one will touch the other alſo, and the points of contact will be diametrically oppoſite to each other on the ſurface of the ſphere.

Let BQP, LWH be two equal and parallel circles of the ſphere, and aBi be a great circle touching one of them in B; I ſay, it will touch the other LWH in a point which is the oppoſite extremity of a diameter of the ſphere drawn through B. *Fig.* 20.

Through C the centre of the ſphere draw BC, and OCU at right angles to the parallel circles BQP, LWH, and let the plane BCO cutting the ſphere in the great circle EID, fall upon the circumference of the circle LWH in the point H oppoſite to B. The points B, C, H are in one ſtraight line (2. Cor. Prop. XIV.), wherefore the circumference of the circle aBi does meet that of the circle LWH in a point H which is diametrically oppoſite to B. But they touch in the point of concourſe H, for becauſe OU is perpendicular to the plane BQP, the great circle EID which paſſes through OU is perpendicular to the plane BQP (18. E. 11.), and becauſe it paſſes alſo through the point of contact B of the circles aBi, BQP, it will be likewiſe perpendicular to the circle aBi (PROP. XII.). But the great circle EID is perpendicular to the circle LWH, wherefore the circles aBi, LWH meeting each other in a point H, and the great circle EID which paſſes through the point of their concourſe being perpendicular to each of them, the circles touch each other in the point H (PROP. XIII.).

PROP. XVI. THEOREM.

IF there be a line on the ſurface of the ſphere, ſuch that all right lines drawn thereto from a certain point alſo on the ſurface, but not in the ſaid line, be equal between themſelves, that line ſhall be the circumference of a circle, and the point, from which the equal right lines are drawn, ſhall be one of its poles.

CASE 1. WHEN a right line, drawn from any point of the line on the ſurface, at right angles to the diameter of the ſphere drawn through the given point on the ſurface, meets the diameter in the centre of the ſphere.

Let EID be a line on the ſurface of the ſphere, ſuch that right lines drawn from a point a in the ſurface to every point in EID be equal between themſelves; I ſay, EID ſhall be the circumference of a circle. Draw aCi through C the centre of the ſphere, and from any point E in the line EID let a perpendicular be drawn to ai, meeting it in C, alſo to any points B, R in the line EID draw the right lines aB, aR, CB, CR, and join aE. Becauſe aE, aB, aR are, by ſuppoſition, equal, as are alſo CE, CB, CR (1. Cor. Def.); and aC is common to each of the triangles aCE, aCB, aCR, therefore in each of them the angles ſubtended by equal ſides are equal (8. E. 1.), viz. the angles aCE, aCB, aCR are all equal between themſelves. But aCE is a right angle, therefore aCB, aCR are alſo right angles. Wherefore the right line aC being at right angles to each of the three ſtraight lines CE, CB, CR at their common point of concourſe C, theſe three ſtraight lines are in one plane (5. E. 11.). By the ſame reaſoning are right lines drawn from every point in the line EID to the centre C ſhewn to be in the ſame plane with any two of the right lines CE, CB, CR, viz. the line is wholly ſituate in one and the ſame plane paſſing through the centre C of the ſphere, and therefore is the circumfe- *Fig.* 20.

rence of a great circle (PROP. I. and 6. DEF.). And becauſe aC is perpendicular to CE, CB, it will be perpendicular to the plane paſſing through CE, CB (4. E. 11.), viz. to the circle EID, and therefore a is one of its poles (8. DEF.)

CASE 2. WHEN the perpendicular to the diameter does not meet it in the centre of the ſphere.

LET BQP be any line on the ſurface of the ſphere, ſuch that right lines drawn thereto from a point R on the ſurface, but not in the line itſelf, are all equal; I ſay, BQP ſhall be the circumference of a circle.

DRAW the diameter RCr, and from any point B in BQP draw BO perpendicular to Rr, meeting it in the point O which is not the centre of the ſphere. Join CB, and to any points T, Q in the line BQP draw CT, CQ, OT, OQ, alſo join RB, RT, RQ. Becauſe CB, CT, CQ, and RB, RT, RQ (HYP.) are equal between themſelves, while CR is common to each of the triangles CBR, CTR, CQR, the angles CRB, CRT, CRQ ſubtended in each by the equal ſides will therefore be equal (8. E. 1.). Again, In the triangles RBO, RTO, RQO, becauſe the angles at the common point R have in each been ſhewn to be equal, and RO being a common ſide, the ſides in each containing theſe equal angles are reſpectively equal, the baſes OB, OT, OQ, and the angles ROB, ROT, ROQ, will be equal between themſelves (4. E. 1.). But the angle ROB is a right angle, therefore the angles ROT, ROQ are alſo right, and ſo RO is perpendicular to the three right lines OB, OT, OQ at their common point of concourſe O; wherefore theſe three right lines are in one plane (5. E. 11.). By the ſame reaſoning is every right line drawn from O to the circumference BQP ſhewn to be in the ſame plane with any two of the right lines OB, OT, OQ, and all theſe right lines to be equal between themſelves; viz. the right line BQP is wholly ſituate in a plane not paſſing through the centre, and therefore is the circumference of a leſs circle, whoſe centre is O (PROP. I. and 7. DEF.). And becauſe ROC a diameter of the ſphere meets OB, OT at right angles, it meets the plane BOT, viz. the circle BQP, at right angles (4. E. 11.), and therefore R is one of its poles (8. DEF.).

COR. IF there be a line on the ſurface of the ſphere, ſuch that the arches of great circles falling thereon from a point on the ſurface of the ſphere, but not in the ſaid line, be equal between themſelves, that line ſhall be the circumference of a circle, and the point through which the great circles falling on the circumference are drawn ſhall be one of the poles thereof.

FOR the great circles being equal, equal arches thereof are ſubtended by equal right lines, therefore the rectilinear diſtances of the point from the line being equal, the line is a circle, and the point one of its poles.

PROP. XVII. THEOREM.

THE arches of parallel circles of the ſphere intercepted between two great circles paſſing through their poles, are ſimilar.

Fig. 22. LET EaD, BTP be two parallel circles of the ſphere, and EBD, GTg be two great circles paſſing through their poles R, r, and falling upon their circumferences in E, B, G, T; I ſay, the arches BT, EG ſhall be ſimilar.

LET Rr the common ſection of the great circles meet the parallels in O, C, and the circles themſelves cut the parallels in the right lines OB, CE, OT, CG. Becauſe OB is parallel to CE, and OT to CG (16. E. 11.), the angle BOT is equal to the angle ECG (10. E. 11.). But the points O, C are the centres of the parallels (1. and 3. COR. PROP. I.), therefore the circumferences BT, EG of the circles EaD, BTP ſubtending equal angles at their centres, will contain equal angles (20. E. 3.), and therefore be ſimilar (LEM. VIII.).

OF THE SPHERE.

BOOK III.

OF SPHERIC ANGLES AND TRIANGLES.

DEFINITIONS.

I. CIRCUMFERENCES or Angles are ſaid to be of the SAME AFFECTION, when they are each leſs, or each greater than a quadrant or right angle; and of DIFFERENT AFFECTION, when the one is leſs, the other greater than a quadrant or right angle.

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II. THE ſum of two circumferences or angles is ſaid alſo to be of the ſame or different affection with the ſum of any other two, when compared in like manner with a ſemicircle or two right angles.

III. IF round the angular point of any right-lined angle a circle be deſcribed, the circumference intercepted by the right lines comprehending the angle is ſaid to be the MEASURE of the angle.

BECAUSE (by LEM. VII.) the intercepted arch has the ſame proportion to the whole circumference of the circle, that the right-lined angle has to four right angles.

IV. EVERY right-lined angle, if applied at the centre of a given circle, is meaſured on the circumference of that circle; and as many parts as the correſponding arch contains of the whole circumference, ſo many parts is the angle ſaid to contain.

V. THIS given circle in ſpheric geometry, is any great circle of the ſphere.

VI. But as every circle contains as many parts as any other, an angle, when meaſured on the circumference of whatever circle, is ſaid to conſiſt of the ſame number of parts, as when it is meaſured on the circumference of another circle.

For the arches of circles ſubtended by the ſame or equal angles, at their centres, are ſimilar to each other (Lem. VIII.).

AXIOM.

THE meaſure of a ſpheric angle is the right-lined angle comprehended under the right lines drawn in the plane of each circle, and touching each other in the point of their concourſe.

This is received as an axiom, which requires no illuſtration, by every writer whom I have met with on the ſubject of ſpheric geometry, though it be far from a ſelf-evident truth. It is not eaſy, perhaps not poſſible, to demonſtrate it, but it may be thus illuſtrated.

Let

Fig. 24. LET the circumferences GP, KP of two great circles of the ſphere meet each other in P, and a plane touching the ſphere in P, let TP, WP be the common ſections of this touching plane with the planes of the great circles, alſo to the centre of the ſphere draw PC. Then TP, WP which are in the planes of the great circles touch the circumferences in P, and are each at right angles to CP the common ſemidiameter of each circle (PROP. X. and IX. B. II.). But Euclid has ſhewn (in 16. EL. III.) that the inclination of the circumference GP to the diameter PC does not differ from the inclination of the touching line TP to PC by any aſſignable right-lined angle. Therefore the inclination of TP to PC is the ſame with that of the circumference GP to PC, and for the ſame reaſon has WP the ſame inclination to PC that the circumference KP has. Wherefore the right lines TP, WP having the ſame inclination to the right line PC that the circumferences GP, KP have, will have the ſame inclination to each other, that the circumferences have to each other; viz. the right-lined angle TPW is equal to the ſpheric angle GPK.

COR. HENCE all ſpheric angles are meaſured on the circumference of a circle, viz. of a great circle; and by the Sine, Tangent, &c. of a ſpheric angle, are underſtood the Sine, Tangent of the arch of a great circle, which is its meaſure.

PROP. I. THEOREM.

A SPHERIC angle is equal to the inclination of the planes of the great circles comprehending it.

FOR, every thing remaining as in the illuſtration of the preceding axiom, becauſe CP is the common ſection of the planes GPC, KPC, and TP, WP are drawn in each plane at right angles to CP, therefore the angle TPW meaſures the inclination of the planes GPC, KPC (6. DEF. E. 11.). But the ſpheric angle GPK is alſo meaſured by the angle TPW (AXIOM.). Therefore the ſpheric angle GPK is equal to the inclination of the planes GPC, KPC, viz. of the great circles comprehending the ſpheric angle. *Fig.* 24.

COR. 1. IF the planes of two great circles be at right angles to each other, their circumferences will alſo be at right angles to each other. And, If the circumferences be at right angles to each other, the planes of the circles will be mutually perpendicular to each other.

FOR, the planes being perpendicular, their inclination to each other, viz. the ſpheric angle comprehended by the circumferences is a right one. And the angle comprehended by the circumferences being a right one, the inclination of the planes is meaſured by a right angle, viz. the planes are perpendicular to each other.

COR. 2. THE vertical or oppoſite angles at the ſame point formed by the interſection of two great circles, are equal between themſelves.

FOR they are equal to the vertical angles comprehended by the ſame tangents at the point of interſection, or they are each equal to the inclination of the planes.

COR. 3. FOR the ſame reaſon therefore the ſpheric angles at the oppoſite interſections of two great circles, and comprehended by the ſame circumferences, are equal between themſelves.

COR. 4. THE adjacent angles at the interſection of two great circles, are together equal to two right angles.

FOR they are equal to the right-lined angles comprehended by the tangents at the point of interſection, which being adjacent are together equal to two right angles.

COR. 5. AND if two great circles meeting each other, make the adjacent angles equal between themſelves, the circles will be at right angles to each other.

FOR the adjacent angles being together equal to two right angles, and alſo equal between themſelves, each of them is a right angle.

COR. 6. HENCE the angle comprehended by the circumferences of two great circles is meaſured by the diſtance of their poles, or by the arch, intercepted between their circumferences, of a great circle deſcribed about the angular interſection as a pole, or by half the ſum or difference of the diſtances of the pole of either of them from the circumference of the other, theſe diſtances being meaſured on the circumference of a great circle deſcribed through the poles of each.

FOR each of theſe three meaſure the inclination of the planes of the circles (PROP. VI. with COR. 1. and 3. B. II.).

PROP. II. THEOREM.

IF the angular points of a ſpheric triangle be the poles of three great circles, another triangle will be formed by their interſection, which is called SUPPLEMENTAL of the former; becauſe the ſides of the ſupplemental triangle will be the ſupplements of the meaſures of the angles of the former triangle, and the meaſures of the angles of the ſupplemental triangle will be the ſupplements of the ſides of the former triangle.

Fig. 32. LET ABC be a ſpheric triangle, and bc, ab, ac be three great circles, whoſe poles are A, C, B, forming by their interſections the ſpheric triangle abc; I ſay, bc, ac, ab ſhall be ſupplemental to the meaſures of the angles A, B, C, and the meaſures of the angles a, b, c ſhall be ſupplemental to the ſides BC, AC, AB.

LET ab, ac meet again in *a*, and AB, BC in P. Becauſe the poles of the great circles ab, ac are C, B, they will each be perpendicular to the great circle CB (PROP. III. B. II.), and therefore each paſs through the poles of BC, wherefore their interſections a, *a* are the poles of BC. For the ſame reaſon is b a pole of AC, and c a pole of AB.

LET bc meet AB in F, K, and AC in G. Becauſe b, c are the poles of AC, AB, thereforefore bG, cK are each quadrants (3. Cor. Prop. IV. B. II.), and equal between themſelves. Wherefore taking away the common arch cG, the arch bc is equal to the arch GK. But becauſe A is the pole of bc, FG is the meaſure of the angle BAC (6. Cor. Prop. I.), and K, F being the oppoſite interſections of two great circles, the arch FGK is a ſemicircle (Prop. II. B. II.), and GK is the ſupplement of FG. Wherefore GK or the equal arch bc is the ſupplement of the arch which meaſures the angle BAC. And by the ſame reaſoning are ac, ab ſhewn to be ſupplemental to the arches which meaſure the angles ABC, ACB.

Let BC meet ac, ab in E, L. Becauſe AB, BC paſs through B a pole of the circle ac, the oppoſite interſection P is the other pole of ac (Prop. III. B. II.), and C is a pole of ab, wherefore PE, CL are each quadrants, and equal between themſelves. Take away or add the arch CE, which is common, and the remaining or the whole arch PC will be equal to the arch EL. But becauſe a is the pole of the circle BC, the angle bac is meaſured by the arch EL, and BCP being a ſemicircle, PC is the ſupplement of BC. Wherefore the meaſure of the angle bac is equal to the arch which is the ſupplement of the ſide BC. And by the ſame reaſoning are the meaſures of the angles abc, acb ſhewn to be equal to the arches which are the ſupplements of the ſides AC, AB.

Cor. If through the points A, a, or A, *a* a great circle be drawn, meeting the circles BC, bc in D, d, this great circle will be perpendicular to BC, bc, becauſe it paſſes through a pole of each, and the perpendicular ad will be the ſupplement of the perpendicular AD, the ſegments cd, bd will be the complements of the meaſures of the angles BAD, CAD, and the meaſures of the angles cad, bad will be the complements of the ſegments BD, CD.

Because *a*, A are the poles of the great circles BC, bc, therefore *a*D, Ad are each a quadrant, and equal between themſelves; conſequently, taking away the common arch Dd, the remaining arch *a*d is equal to the remaining arch AD. But ad is the ſupplement of *a*d, becauſe ad*a* is a ſemicircle, therefore ad is the ſupplement of an arch equal to AD.

And becauſe c is the pole of AB, cF is a quadrant, and cd is the complement of dF. But dF is the meaſure of the angle BAD, becauſe A is the pole of the circle bc; therefore cd is the complement of the meaſure of the angle BAD. By the like reaſoning is bd ſhewn to be the complement of the meaſure of the angle CAD.

Also, Becauſe B is the pole of the circle ac, BE is a quadrant, viz. ED is the complement of BD. But ED meaſures the angle cad, therefore the meaſure of the angle cad is the complement of BD. For the ſame reaſon, is the meaſure of the angle bad the complement of CD.

PROP. III. THEOREM.

THE ſame things remaining, if the three ſides of the ſupplemental triangle be continued to meet each other in *a*, *b*, *c*, three triangles will be formed thereby, viz. *a*bc, *b*ac, *c*ab, two ſides in each of which are equal to the meaſures of two angles, but the remaining ſide is the ſupplement of the remaining angle of the triangle ABC. Alſo the angles in each, oppoſite to the two ſides, are meaſured by the correſpondent ſides of the triangle ABC, but the meaſure of the remaining angle in each will be equal to the ſupplement of the remaining ſide of the triangle ABC. Wherefore theſe triangles are called SEMI-SUPPLEMENTAL.

LET *a*c meet the circle AB in I; becauſe *a*, c are the poles of BC, AB, the arches *a*E, cI will be quadrants, wherefore *Fig.* 32.

wherefore taking away from each the common arch cE, the arch *a*c is equal to the arch EI. But B being the pole of *a*c, the angle ABC is meaſured by the arch EI, therefore the arch *a*c is equal to the meaſure of the angle ABC. For the ſame reaſon, is the ſide *a*b equal to the meaſure of the angle ACB. But the remaining ſide bc is ſupplemental to the meaſure of the remaining angle BAC (PROP. II.).

LET AB meet the circle *a*c in O. Becauſe B, A are poles of the circles *a*c, bc, the arches BO, AF are quadrants, wherefore taking away the common arch BF, the arch FO is equal to the arch AB. But FO is the meaſure of the angle *a*cb, therefore the meaſure of the angle *a*cb is equal to the ſide AB. For the ſame reaſon, is the meaſure of the angle *a*bc equal to the ſide AC. But the remaining angle b*ac* is equal to the oppoſite angle bac (3. COR. PROP. I.), and the meaſure of the angle bac is ſupplemental to the ſide BC (PROP. II.).

BY the ſame reaſoning is it ſhewn that the ſides a*c*, b*c* of the triangle *c*ab meaſure the angles B, A, and the angles ab*c*, ba*c* are meaſured by the ſides AC, BC, but that the ſide ab is ſupplemental to the meaſure of the angle C, and the meaſure of the angle a*c*b ſupplemental to the ſide AB. And the like of the triangle *b*ac.

Cor. If in one of the ſemi-ſupplemental triangles, as *a*bc, a great circle *a*d be drawn through the angle *a* at at right angles to the baſe bc, viz. the angle whoſe meaſure is the ſupplement of BC, and through the angle A a great circle AD be drawn at right angles to BC, the perpendicular arches *a*d, AD will be equal between themſelves, the ſegments cd, bd will be the complements of the meaſures of the angles BAD, CAD, and the meaſures of the angles c*a*d, b*a*d will be the complements of the ſegments BD, CD.

This Corollary is comprehended in the Corollary of the preceding Propoſition, for it was there ſhewn that the perpendicular arches *a*d, AD are portions of one and the ſame great circle, and that they are equal portions. Alſo, that cd, bd are complements of the meaſures of the angles BAD, CAD, and becauſe the angles c*a*d, b*a*d are equal to the oppoſite angles cad, bad, that therefore the meaſures of the angles c*a*d, b*a*d are the complements of the ſegments BD, CD.

PROP. IV. THEOREM.

If two ſides of a ſpheric triangle be equal between themſelves, the angles ſubtended by them ſhall alſo be equal; And, If two angles of a ſpheric triangle be equal, the ſides ſubtending them ſhall likewiſe be equal.

Fig. 25. LET DAB be a ſpheric triangle, whoſe ſides AD, DB are equal between themſelves; I ſay, the angles DBA, DAB ſhall alſo be equal.

To C the centre of the ſphere draw AC, BC, DC, and in the plane of the baſe AB draw AK, BL perpendicular to AC, BC, and join AB. Becauſe the angles CAK, CBL are together equal to two right angles, the angles BAK, ABL are together leſs than two right angles, and therefore AK, BL do meet. Let them meet in E, and in the plane of the circle AD draw AF at right angles to CA meeting CD in F; alſo join FB, FE, CE. Becauſe the arches AD, BD of equal circles, are equal between themſelves, the angles ACD, BCD at the centre of each ſubtended by them are equal (27. E. 3.), and the ſides AC, CF are equal to the ſides BC, CF, each to each, therefore the baſes AF, BF are equal (4. E. 1.). And becauſe the two ſides AC, CE are equal to the two BC, CE, and of the homologous angles, two EAC, EBC are equal being right, and therefore the two AEC, BEC are each leſs than a right angle, the baſes AE, BE will be equal (LEM. I.). Wherefore the two ſides EA, AF being equal to the two EB, BF, and the ſide EF common, the angle FAE will be equal to the angle FBE (8. E. 1.). But becauſe FA, EA are perpendicular to AC, they touch the circles AD, AB in their interſection A (Cor. 16. E. 3.); wherefore the angle FAE meaſures the ſpheric angle DAB (Ax.). For the ſame reaſon, the angle FBE meaſures the ſpheric angle DBA, and conſequently the ſpheric angles themſelves are equal.

And Converſely, If the angles DAB, DBA of a ſpheric triangle be equal between themſelves, the ſides DB, DA ſubtending them ſhall be equal.

Let dab be the ſemi-ſupplemental triangle, whoſe ſides ad, bd meaſure the angles DBA, DAB, and whoſe angles dba, dab at the baſe ab are meaſured by the ſides AD, BD (Prop. III.). Becauſe therefore the angles DBA, DAB are equal, the ſides ad, bd which meaſure them are equal, and conſequently the angles dba, dab, ſubtended by theſe equal ſides, are alſo equal. But the ſides AD, BD of the triangle ABD meaſure the angles dba, dab, therefore the ſides AD, BD are equal between themſelves. *Fig.* 33.

Cor. Hence every Equilateral ſpheric triangle is alſo Equiangular; and every Equiangular ſpheric triangle is alſo Equilateral.

PROP. V. THEOREM.

IF the three ſides of a ſpheric triangle be equal to the three ſides of another, each to each, the angles of the one ſhall be equal to the correſpondent angles of the other.

Fig. 35. 36. 37. LET the ſpheric triangles ABC, ABD have the ſide AB common, or equal in each,* and the ſides AC, CB equal to the ſides AD, DB, each to each; I ſay, the angles ſubtended by equal ſides in each ſhall be equal between themſelves.

IF the triangles have not the ſide AB common to each, the one may be made, or conceived, to approach the other, ſo that two equal ſides ſhall coincide, and become a common ſide, and the triangles be ſituated towards different parts of this common ſide AB.* Then either the ſides

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* No greater aſſumption is made here than by Euclid (in the IV. EL. I.), for though in neither caſe the mere abſtract triangles can be removed, without removing a real ſolid body on whoſe ſuperficies they are deſcribed, yet it is ſufficient to the purpoſe, if the mind do but eaſily admit the conception of ſuch a removal. Since it is indifferent with reſpect to the magnitude of the triangle and all its parts, whether they be ſituated in this or that part of the plane or ſpheric ſuperficies on which they are deſcribed.

CB, BD are in the ſame great circle, or they are not. If they be, ſince in the triangle ACD the ſide AC is equal to the ſide AD, the angle ADC is equal to the angle ACD (Prop. IV.), viz. the correſpondent angles ADB, ACB of the triangles ABC, ABD, are equal. And in like manner, by applying the triangles, ſo as the other equal ſides coincide, is it ſhewn that the other correſpondent angles are equal. *Fig.* 37.

But if the ſides CB, BD be not in the ſame great circle, let CD be the great circle paſſing through the points C, D. Becauſe AC is equal to AD, the angle ACD is equal to the angle ADC, and becauſe BC is equal to BD, the angle BCD is equal to the angle BDC. Wherefore the whole or remaining angle ACB is equal to the whole or remaining angle ADB. And by the ſame reaſoning is it ſhewn that the angles ABC, BAC are equal to the correſpondent angles ABD, BAD. *Fig.* 35. 36.

PROP. VI. THEOREM.

IF the three angles of a ſpheric triangle be equal to the three angles of another ſpheric triangle, the correſpondent ſides ſhall alſo be equal.

Fig. 34. LET ABC, DEF be two ſpheric triangles, in which the angles A, B, C are equal to the angles D, E, F, each to each; I ſay, the correſpondent ſides AB, DE, and AC, DF, and BC, EF ſhall alſo be equal.

LET abc, def be the ſemi-ſupplemental triangles. Becauſe the angles B, C are reſpectively equal to the angles E, F, the ſides ac, ab, which are the meaſures of the former, ſhall be reſpectively equal to the ſides df, de, which are the meaſures of the latter. And becauſe the angle A is equal to the angle D, the ſide bc, which is the ſupplement of the meaſure of the former, ſhall be equal to the ſide ef, which is the ſupplement of the meaſure of the latter. Wherefore the three ſides of the triangle abc being equal to the three ſides of the triangle def, the correſpondent angles in each are equal (PROP. V.). The ſides AB, DE, therefore, which are the meaſures of the equal angles c, f, ſhall be equal, as for the ſame reaſon are the ſides AC, DF. And alſo the ſides BC, EF, which are the ſupplements of the meaſures of the equal angles a, d, ſhall be equal between themſelves.

PROP. VII. THEOREM.

IF a great circle paſſing through the vertex of an iſoſceles ſpheric triangle biſect the baſe, it will alſo biſect the vertical angle, and meet the baſe at right angles. And Converſely, If it biſect the vertical angle, it will alſo biſect the baſe, and meet it at right angles.

LET ADC be an iſoſceles ſpheric triangle, having the ſide AD equal to the ſide AC; if a great circle AB paſſing through the vertical angle A biſect the baſe DC in B, it will alſo biſect the vertical angle DAC, and meet the baſe DC at right angles. *Fig.* 37.

BECAUSE BD is equal to BC, AB common to the two triangles ADB, ACB, and the ſide AD equal to the ſide AC, the angle DAB ſhall be equal to the angle CAB, as alſo the angle ABD equal to the angle ABC (PROP. V.). But if the adjacent angles made by the concourſe of two great circles be equal between themſelves, the circles ſhall be at right angles to each other (5. COR. PROP. I.). Wherefore AB is perpendicular to DC.

AND Converſely, If AB biſect the vertical angle DAC, it ſhall meet the baſe DC at right angles, and biſect it in the concourſe B.

For if DC be not biſected in B, let F be the point in which it is biſected, and AF be the great circle paſſing through A, F. Then becauſe the great circle AF biſects the baſe DC, it will alſo biſect the vertical angle DAC. But AB, by hypotheſis, does alſo biſect the ſame angle, therefore the angles DAB, DAF, being each the halves of one and the ſame magnitude, are equal between themſelves, viz. the whole is equal to its part, which is abſurd. Wherefore DC is not biſected in F, nor, for the ſame reaſon, in any point unleſs B; and conſequently AB is alſo perpendicular to DC.

PROP. VIII. THEOREM.

If the vertex of an iſoſceles ſpheric triangle be not the pole of the baſe, a great circle paſſing through the vertex at right angles to the baſe, will biſect both the baſe and vertical angle.

Fig. 37. The ſame things remaining, if the vertex A be not the pole of the baſe DC, the vertical angle DAC and the baſe DC will each be biſected by the great circle AB meeting the baſe at right angles.

BECAUSE AB, being perpendicular to DC, paſſes through the poles of DC (PROP. III. B. II.), let P be one of its poles in the great circle AB, and PD, PC be the great circles paſſing through P and the points D, C. In the triangles DPA, CPA, the ſide AD is equal to the ſide AC, AP is common, and the remaining ſides PD, PC are equal (1. COR. PROP. IV. B. II.), therefore the angles DPA, CPA are equal between themſelves (PROP. V.). Wherefore becauſe the great circle PAB biſects the vertical angle of the iſoſceles triangle DPC, it will alſo biſect the baſe (PROP. VII.); but the ſame being the baſe of the iſoſceles triangle DAC, the ſame great circle does therefore biſect the vertical angle of the triangle DAC.

PROP. IX. THEOREM.

IF two ſides of a ſpheric triangle be equal to two ſides of another ſpheric triangle, each to each, and the angles, comprehended by the equal ſides in each, be equal, the remaining ſides, and the remaining correſpondent angles ſhall be equal.

LET the triangles as in PROP. V. approach each other, ſo as two equal ſides become one common ſide AB, two other *Fig.* 35. 36. 37.

other ſides AC, AD being equal between themſelves, and the angle CAB equal to the angle DAB; I ſay, the baſe CB ſhall be equal to the baſe DB, and the remaining correſpondent angles ACB, ADB, and ABC, ABD ſhall be equal.

Fig. 37. IF the baſes CB, BD be in the ſame great circle, then becauſe the vertical angle of the iſoſceles triangle ACD is biſected by the great circle AB, the baſe CD is biſected alſo, viz. CB is equal to BD, the angles ABC, ABD are right, and therefore equal (PROP. VII.); and the triangle being iſoſceles, the angle ACB is equal to the angle ADB (PROP. IV.).

Fig. 35. 36. 37. BUT if the baſes CB, BD be not in the ſame great circle, let CD be the great circle paſſing through the points C, D. If B be the pole of the circle CD, BC, BD are each quadrants, and conſequently equal between themſelves. Wherefore the remaining correſpondent angles are equal between themſelves (PROP. V.).

IF B be not the pole of CD, let E be the interſection of the great circles AB, CD. Becauſe AC is equal to AD, and the angle CAE to the angle DAE, the triangle ACD is iſoſceles, and its vertical angle is biſected by the great circle AE. Wherefore AE meets the baſe at right angles, and biſects it in E (Prop. VII.). Becauſe the great circle ABE, therefore, is at right angles to the circle CD, it paſſes through the poles of CD (Prop. III. B. II.), and becauſe from a point B therein which is not either of the poles, are deſcribed the great circles BC, BD intercepting the equal arches EC, ED, the arches BC, BD ſhall themſelves be equal (2. Cor.* Prop. V. B. II.). Wherefore the three ſides of the ſpheric triangles ABC, ABD being equal between themſelves, each to each, the correſpondent angles ſhall be equal (PROP. V.).

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* This Corollary, which was intended to have been the ſubſequent Prop. to the V. B. II., was by accident omitted in the manuſcript, while the ſheet which ought to have contained it was under the printer's hands, and therefore could not be inſerted in its proper place. But it is eaſily thus demonſtrated.

The ſame things remaining as in Prop. V. B. II., viz. the great circle BIP being perpendicular to a circle BQP of the ſphere, and cutting it in the diameter BP, through a point I therein which is not the pole of BQP, let two great circles IT, IG be deſcribed, intercepting equal arches BT, BG; I ſay, the circumferences IT, IG ſhall alſo be equal. For join the right lines OT, OG, ST, SG, IT, IG. Becauſe the arches BT, BG are equal, the angles BOT, BOG ſhall be equal (27. E. 3.), and conſequently alſo the angles TOS, GOS. Wherefore the ſide TO being equal to the ſide GO, OS common, and the angles TOS, GOS equal between themſelves, the baſe TS is equal to the baſe GS (4. E. 1.). Again, Becauſe TS is equal to GS, IS common, and the angle TSI equal to the angle GSI, becauſe each is a right angle, the baſe TI ſhall be equal to the baſe GI. Wherefore the arches IT, IG, of equal circles, being ſubtended by equal right lines, are themſelves equal (28. E. 3.). *Fig.* 20.

PROP. X. THEOREM.

IF two ſpheric triangles have two angles of the one equal to two of the other, each to each, and the ſides included between theſe equal angles be alſo equal; the remaining angle of the one ſhall be equal to the remaining angle of the other, and the remaining ſides, taken correſpondently, ſhall be equal.

Fig. 34. LET the ſpheric triangles ABC, DEF have two angles B, C of the one equal to two E, F of the other, and the ſide BC of the one equal to the ſide EF of the other; I ſay, the remaining angles A, D, and the remaining correſpondent ſides AB, DE, and AC, DF ſhall be equal.

LET abc, def be the ſemi-ſupplemental triangles. Becauſe the angles B, E are equal between themſelves, the ſides ac, df, which are the meaſures of them, are alſo equal. For the ſame reaſon, are the ſides ab, de equal between themſelves. And becauſe the ſides BC, EF are equal, the angles a, d, whoſe ſupplements are meaſured by them, will be equal. Wherefore two ſides ab, ac of the ſpheric triangle abc being equal to two ſides de, df of the triangle def, and the angle bac equal to the angle edf, the baſe bc ſhall be equal to the baſe ef, and the correſpondent angles b, e, and c, f ſhall be equal (Prop. IX.). Therefore the ſides AB, DE, and AC, DF, which are the meaſures of the equal angles c, f, and b, e, as alſo the angles A, D, which are meaſured by the ſupplements of the equal ſides bc, ef, will be ſeverally equal.

PROP. XI. THEOREM.

If two ſpheric triangles have two ſides of the one equal to two of the other, each to each, but no one of the equal ſides be a quadrant, and of the homologous angles, if one in each be a right angle; the baſes and the remaining homologous angles ſhall be equal.

Let the ſpheric triangles ABC, ABD, as in Prop. V., approach each other, ſo as two equal ſides become one common ſide AB, two other ſides AC, AD being equal to each other, and let the angles ABC, ABD, oppoſite to two of *Fig.* 37.

of the equal ſides, be each a right angle; I ſay, the baſes BC, BD, and the remaining angles, viz. oppoſite to equal ſides, ſhall alſo be equal.

THE angles ABC, ABD, being each a right angle, are together equal to two right angles, and they are adjacent angles; therefore the baſes BC, BD are in one and the ſame great circle (COR. PROP. I.), and the triangle ACD is iſoſceles. But becauſe AC, AD are not quadrants, A is not the pole of CD (3. COR. PROP. IV. B. II.), wherefore the great circle AB, which is perpendicular to the baſe CD, biſects the baſe, as alſo the vertical angle CAD (PROP. VIII.). The baſe BC is, therefore, equal to the baſe BD, the angle BAC to the angle BAD, and the angle ACB to the angle ADB (PROP. IV.).

COR. IF two ſpheric triangles have two angles of the one equal to two of the other, each to each, and of the homologous ſides, if one in each be a quadrant, the two remaining angles, and the remaining homologous ſides ſhall be equal.

Fig. 34. LET ABC, DEF be two ſpheric triangles, having two angles B, C of the one equal to two E, F of the other, and if two ſides AB, DE, oppoſite to equal angles, be each a quadrant; the angles A, D, and the ſides AC, DF, as alſo

BC, EF ſhall be equal. Let abc, def be the ſemi-ſupplemental triangles. Becauſe the angles B, C are equal to the angles E, F, the arches which are the meaſures of them will be equal, viz. the ſides ac, ab are equal to the ſides df, de, and becauſe AB, DE are each a quadrant, the angles c, f, which are meaſured by them, ſhall be each a right angle. Wherefore the baſes bc, ef, and the remaining homologous angles of the triangles abc, def are equal between themſelves. Conſequently the correſpondent homologous parts of the triangles ABC, DEF will alſo be equal.

PROP. XII. THEOREM.

IF two ſpheric triangles have two ſides of the one equal to two of the other, each to each, and of the homologous angles, if two be equal between themſelves, but the other two of like affection, viz. each greater, or each leſs than a right angle; the remaining ſides, and the remaining angles, as alſo the two homologous ones of like affection, ſhall be equal between themſelves.

LET

Fig. 35. 36. Let the ſpheric triangles ACB, ADB, as in Prop. V., approach each other, ſo as two equal ſides become one common ſide AB, two other ſides AC, AD being equal to each other, while of the homologous angles, two ACB, ADB are equal between themſelves, but the other two ABC, ABD are either both greater or leſs than right angles; I ſay, the remaining ſides BC, BD, the angles ABC, ABD, and the angles CAB, DAB ſhall be equal.

Because each of the angles ABC, ABD is greater or leſs than a right angle, they are together greater, or together leſs, than two right angles. Wherefore CB, BD are not in the ſame great circle (4. Cor. Prop. I.). Let CD be the great circle paſſing through C, D. Becauſe AC is equal to AD, the angle ACD is equal to the angle ADC (Prop. IV.), but the angle ACB is equal to the angle ADB (Hyp.), therefore the remaining angle BCD is equal to the remaining angle BDC, and conſequently in the triangle CBD the ſide BC is equal to the ſide BD (Prop. IV.). Wherefore in the triangles ABC, ABD, the three ſides of the one being equal to the three ſides of the other, each to each, the homologous angles CAB, DAB, and ABC, ABD are equal between themſelves (Prop. V.).

Cor. If two ſpheric triangles have two angles of the one equal to two of the other, each to each, and if two of the homologous ſides be each greater or leſs than a quadrant, but the other two equal between themſelves; the remaining angles, the remaining ſides, and the two homologous ones of like affection ſhall alſo be equal.

THIS is inferred almoſt in the ſame words as the Corollary of the preceding Propoſition, with this one additional circumſtance, that two ſides of the original triangles being each greater or leſs than a quadrant, the correſponding angles of the ſemi-ſupplemental triangles are therefore each greater or leſs than a right angle. Wherefore the ſemi-ſupplemental triangles will have all the conditions required in this Propoſition.

PROP. XIII. THEOREM.

IN a ſpheric triangle any two ſides are greater than the third.

IF the triangle be equilateral, then any two ſides are double to the third.

WHEREFORE let RIQ be a ſpheric triangle, which is not equilateral, and of any two unequal ſides RI, RQ let RI be the leſs. Draw RCr the diameter of the ſphere, which will be in the plane of the great circle RI (1. COR. PROP. I. B. II.), draw QO at right angles *Fig.* 20.

angles to Rr, and in the plane of RI draw POB at right angles to Rr. Becauſe Rr a diameter of the ſphere is at right angles to OB, OQ, it will be at right angles to the plane paſſing through OB, OQ (4. E. 11.). Wherefore R is the pole, and O the centre of the circle, which is the ſection of the plane BOQ with the ſphere (2. COR. PROP. I.). The circumferences RB, RQ, RP are, therefore, equal between themſelves (1. COR. PROP. IV. B. II.). But becauſe RI is leſs than RQ, viz. than RP, the point I is between R and P, and the arch IB is equal (to the arches IR, RB, viz.) to the arches IR, RQ. But IB is greater than IQ (COR. PROP. V. B. II.), therefore the two ſides IR, RQ of the ſpheric triangle RIQ are greater than the third ſide IQ.

BY the ſame reaſoning, becauſe IQ is greater than IP, therefore IQ, IR together are greater (than RP, viz.) than RQ.

AND becauſe RQ is, by ſuppoſition, greater than RI, therefore RQ, QI are much greater than RI.

OTHERWISE.

Fig. 20. LET aBE be a triangle on the ſurface of the ſphere, whoſe planes paſſing through the centre of the ſphere, form there a ſolid angle, contained under the three plane angles aCE, ECB, BCa. But theſe plane angles are the meaſures of the ſpheric ſides aE, EB, Ba (3. DEF.), and any two plane angles conſtituting a ſolid angle are greater than the third (20. E. 11.). Therefore any two of the three ſpheric ſides aE, EB, Ba are greater than the third.

COR. THE circumference of a great circle deſcribed between any two points on the ſurface of the ſphere, is leſs than any line whatever deſcribed on the ſurface between the ſame points.

LET B, P be any two points on the ſurface of the ſphere, and BIP be the circumference of a great circle, alſo BQP any other line deſcribed between the points B, P, and on the ſurface of the ſphere; I ſay, BIP is leſs than BQP. Conceive the line BQP to be divided into the parts BT, TQ, QP, and great circles to be deſcribed through B, T, T, Q, Q, P, and P, T. Then in the ſpheric triangles BPT, TPQ, the circumference BIP is leſs than the circumferences BT, TP, and TP itſelf is leſs than the circumferences TQ, QP, wherefore the circumference BIP is much leſs than the circumferences BT, TQ, QP. And the ſame concluſion would be obtained, into whatever number of parts the line BQP had been divided, but as *Fig.* 20.

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this number is increaſed, the points of the diviſion approach nearer and nearer to each other, and the circumferences of the great circles deſcribed through every two ſucceſſive points approach nearer to an equality with the correſponding parts of the line BQP, and at length, when the number is increaſed indefinitely, all the circumferences deſcribed in ſucceſſion muſt be deemed to coincide with, and conſtitute the line BQP itſelf. But the circumference BIP is leſs than all theſe circumferences taken together, and therefore is leſs than the line BQP.

PROP. XIV. THEOREM.

IN a ſpheric triangle the greater ſide ſubtends the greater angle, and the greater angle the greater ſide.

Fig. 38. LET ABC be a ſpheric triangle, in which the angle BAC is greater than the angle ABC; I ſay, the ſide BC oppoſite to the greater angle is greater than AC which is oppoſite to the leſs.

SINCE the angle BAC is greater than the angle ABC, a a great circle making with AB at the point A an angle equal to ABC will fall within the triangle. Let the great circle AD be thus drawn meeting BC in the point D. Becauſe the angle DBA is equal to the angle DAB, the ſide DA is equal to the ſide BD (Prop. IV.), wherefore adding DC which is common, the whole BC is equal to AD, DC together. But AD, DC together are greater than AC (Prop. XIII.), therefore BC is greater than AC.

And if in a ſpheric triangle ABC a ſide BC be greater than the ſide AC, the angle BAC oppoſite to the greater ſhall be greater than the angle ABC oppoſite to the leſs ſide. For the angle at A is not equal to the angle at B, becauſe then the ſide BC would be equal to the ſide AC (Prop. IV.), which is contrary to the hypotheſis; neither is it leſs, ſince BC would then be leſs than AC, which is alſo contrary to the hypotheſis. Wherefore the angle BAC muſt be greater than the angle ABC.

PROP. XV. THEOREM.

THE three ſides of any ſpheric triangle are together leſs than a great circle.

Let ABC be a ſpheric triangle; I ſay, the three ſides are leſs than a great circle. Let E be the other interſection of the circles BA, BC. Becauſe BAE, BCE are each ſemicircles *Fig.* 38.

circles (PROP. II. B. II.), they are together equal to a great circle; but in the triangle ACE, the ſide AC is leſs than the ſides AE, EC (PROP. XIII.), wherefore adding the common arches BA, BC, the three BA, BC, CA are leſs than the circumferences BAE, BCE, viz. than a great circle.

PROP. XVI. THEOREM.

THE three angles of a ſpheric triangle are greater than two right angles, and leſs than ſix.

Fig. 32. LET ABC be a ſpheric triangle, and abc that which is ſupplemental to it. Becauſe each angle of the triangle ABC together with the meaſure of the correſpondent ſide of the triangle abc is equal to two right angles (PROP. II.), therefore the three angles of the triangle ABC together with the meaſures of the three ſides of the triangle abc are equal to ſix right angles. But the three ſides of the triangle abc are leſs than a great circle, or their meaſures than four right angles (PROP. XV.), therefore the three angles of the triangle ABC are alone greater than two right angles, but leſs than ſix.

COR. 1. THE exterior angle of a ſpheric triangle (made by producing one of its ſides) is leſs than the two interior and oppoſite ones.

FOR the external angle together with the internal adjacent one (being equal to two right angles) is leſs than the three internal angles of the triangle, and therefore the external one alone is leſs than the two internal and oppoſite ones.

COR. 2. IN a right-lined ſpheric triangle, the two oblique angles are together greater than one right angle, but leſs than three.

PROP. XVII. THEOREM.

IN a ſpheric triangle as any two ſides are equal to, greater, or leſs than a ſemicircle, an internal angle at the baſe will be equal to, greater, or leſs than the oppoſite external one at the baſe.

IN the ſpheric triangle ABC, if one ſide AB be produced to F, then if the ſides AC, CB be equal to a ſemicircle, I ſay the angle CAF ſhall be equal to the angle ABC. Let *Fig.* 38.

E

E be the oppoſite interſection of the circles BA, BC, becauſe therefore AC, CB are equal to a ſemicircle, viz. to BCE, take away BC which is common, and AC ſhall be equal to CE. Wherefore the angle CAE is equal to the angle CEA (PROP. IV.), viz. to the angle ABC (3. COR. PROP. I.).

Fig. 40. AGAIN, If AC, CB be greater than a ſemicircle, the angle ABC ſhall be greater than the angle CAE. For AC, CB being greater than a ſemicircle, viz. than BCE, AC will be greater than CE, and therefore the angle (CEA, or) ABC will be greater than the angle CAE (PROP. XIV.).

Fig. 39. LASTLY, If AC, CB be leſs than a ſemicircle, the angle ABC ſhall be leſs than the angle CAE. For by the ſame reaſoning, AC will be leſs than CE, and therefore the angle (CEA, or) ABC will be leſs than the angle CAE.

PROP. XVIII. THEOREM.

ANY two ſides of a ſpheric triangle taken together are of the ſame affection with the oppoſite angles taken together.

LET ABC be a ſpheric triangle; I ſay, any two ſides AC, CB are together of the ſame affection with the oppoſite angles ABC, BAC. *Fig.* 38. 39. 40.

FOR as AC, CB are equal to, greater, or leſs than a ſemicircle, the angle ABC will be equal to, greater, or leſs than the external angle CAE, viz. the angles ABC, CAB together will be equal to, greater, or leſs than the angles CAE, CAB, i. e. than two right angles.

COR. IN an iſoſceles ſpheric triangle, either ſide is of the ſame affection with either angle at the baſe, viz. as the angles at the baſe are each equal to, greater, or leſs than a right angle, each of the ſides will be equal to, greater, or leſs than a quadrant.

PROP. XIX. THEOREM.

IN a right-angled ſpheric triangle, the ſides about the right angle are of the ſame affection with the oppoſite angles.

LET ABC be a ſpheric triangle having a right angle at B; I ſay, that as either ſide BC about the right angle is greater *Fig.* 39. 40.

greater or leſs than a quadrant, the oppoſite angle BAC ſhall be greater or leſs than a right angle.

SINCE the great circles AB, BC are at right angles to each other, they paſs mutually through the poles of each other (1. COR. PROP. III. B. II.), wherefore in the circle BC let D be the pole of AB. As BC is greater or leſs than a quadrant, viz. than BD, the pole D will fall within or without the terms B, C, and accordingly the great circle deſcribed through the points D, A will fall within or without the angle BAC. The angle BAC will therefore be accordingly greater or leſs than the angle BAD. But becauſe D is the pole of AB, DA will be perpendicular to AB (PROP. III. B. II.), viz. DAB will be a right angle; wherefore the angle BAC is greater or leſs than a right angle, accordingly as the oppoſite ſide BC is greater or leſs than a quadrant.

COR. 1. BY the ſame reaſoning is it ſhewn, that if either ſide be a quadrant, the oppoſite angle is a right one.

Fig. 42. FOR the point C would then be the pole of AB, and therefore the angle BAC be a right angle.

Cor. 2. And Converſely, As either of the remaining angles is equal to, greater, or leſs than a right angle, the oppoſite ſide is equal to, greater, or leſs than a quadrant.

For if BAC be alſo a right angle, then CA, CB are each perpendicular to AB, and C will be the pole of AB (3. Cor. Prop. III. B. II.), wherefore BC is a quadrant. But if the angle BAC be greater or leſs than a right angle, then AD will accordingly fall within or without the angle BAC, and therefore BC will be accordingly greater or leſs than BD, viz. than a quadrant. *Fig.* 42. 39. 40.

PROP. XX. THEOREM.

In a right-angled ſpheric triangle, as the ſides about the right angle are each of the ſame affection, or different; the hypothenuſe will be leſs, or greater than a quadrant.

In the triangle ABC right-angled at B, if the ſides AB, BC be each leſs than a quadrant, the hypothenuſe AC ſhall be leſs than a quadrant. Let E be the oppoſite interſection of the circles AB, BC, and becauſe they paſs through the poles of each other, let D, F be a pole of each, and deſcribe 39.

the great circle DA. Becauſe BA is leſs than a quadrant, viz. than BF, the point A is between the interſection B, and the pole F, wherefore AB is the leaſt circumference which can be drawn from A to the circle BC (Cor. Prop. V. B. II.). And becauſe in like manner the point C is between the terms B, D, AC is nearer to AB than AD is, therefore AC is leſs than AD (Cor. Prop. V. B. II.), viz. than a quadrant (3. Cor. Prop. IV. B. II.).

And in the triangle AEC, having a right angle at E, if AE, EC be each greater than quadrants, the points A, C will therefore fall as before between the terms B, F and B, D, and conſequently for the ſame reaſons AC will be leſs than AD or a quadrant.

Fig. 40. But if one ſide AB be leſs, the other ſide BC greater than a quadrant, while A as before is between the terms B, F, the point C will fall on the parts of D remoter from B; therefore AC is remoter from AB than AD is, and conſequently AC is greater than AD, viz. than a quadrant.

Cor. 1. In a right-angled ſpheric triangle, as the hypothenuſe is leſs or greater than a quadrant, the ſides about the right angle will be of the ſame, or different affection.

FOR if the ſides were of different affection, the hypotheнuſe would be greater than a quadrant; if of the ſame affection, it would be leſs: which concluſion in each caſe is contrary to the hypotheſis.

COR. 2. In a right-angled ſpheric triangle, as the hypothenuſe and either ſide are of the ſame or different affection, the other ſide will be leſs or greater than a quadrant.

FOR if the hypothenuſe and either ſide be both leſs than quadrants, the other ſide muſt be leſs alſo, as otherwiſe the hypothenuſe would be greater than a quadrant. And if the hypothenuſe and one ſide be both greater than quadrants, the other ſide muſt ſtill be leſs, or the hypothenuſe would be neceſſarily leſs than a quadrant, which is abſurd. Again, when the hypothenuſe and one ſide are of different affection, let this ſide be leſs than a quadrant, the hypothenuſe therefore is greater; but if the other ſide in this caſe be not greater than a quadrant, the hypothenuſe would alſo be leſs than a quadrant, which is abſurd. Or if the ſide of different affection from the hypothenuſe be greater than a quadrant, then the hypothenuſe is leſs, and it would at the ſame time be required to be greater than a quadrant, if the other ſide be denied to be greater than a quadrant.

PROP. XXI. THEOREM.

THE hypothenuſe of a right-angled ſpheric triangle will be leſs or greater than a quadrant, as the oblique angles are of the ſame or different affection.

For whatever affection the angles be of, the ſides are alike circumſtanced (Prop. XIV.), therefore the Propoſition is demonſtrated in the preceding.

Cor. 1. As the hypothenuſe is leſs or greater than a quadrant, the oblique angles will be of the ſame or different affection.

Cor. 2. As the hypothenuſe and either oblique angle are of the ſame or different affection, the other oblique angle will be leſs or greater than a quadrant.

These two Corollaries are inferred from this Propoſition in the ſame manner as the Corollaries of the preceding.

PROP. XXII. THEOREM.

If through one of the angles of a ſpheric triangle a great circle be drawn perpendicular to the baſe, as this perpendicular falls within or without the triangle, the angles at the baſe will be of the ſame or different affection. And, as the angles at the baſe are of the ſame or different affection, the perpendicular will accordingly fall within or without the triangle.

Let ABC be a ſpheric triangle, from one of whoſe angles A a great circle AD is drawn perpendicular to the baſe BC; if AD falls within the triangle, I ſay the angles at the baſe BC ſhall be of the ſame affection. For, becauſe AD is a ſide common to each of the right angled triangles ABD, ACD, the oblique angles at B, C, ſhall each be of the ſame affection with AD (Prop. XIV.), and therefore of the ſame affection with each other. *Fig.* 42.

But if the perpendicular AD fall without the triangle, then ſhall the angles at B, C be of different affection. For 41.

in the right-angled triangles ACD, ABD, the angles ACD, ABD are of the ſame affection with the common ſide AD, viz. of the ſame affection with each other. But the oblique angles ABC, ABD being together equal to two right angles, whatever affection ABD be of, the angle ABC is of a different affection, therefore the angle ACD or ACB is of a different affection from ABC.

AND Converſely, if the angles at the baſe of a ſpheric triangle be of the ſame affection, the perpendicular from the vertex to the baſe ſhall fall within; but if of different, it ſhall fall without the triangle. For in the firſt caſe, if the perpendicular fall without the triangle, the angles at the baſe would be of different affection; and in the ſecond caſe, if it fall within, the angles would then be of the ſame affection, each of which concluſions is contrary to the hypotheſis.

SCHOL. There are ſome other conſiderations reſpecting a perpendicular to any ſide of a ſpheric triangle from the oppoſite angle, which deſerve ſome notice.

1. IF a great circle DAd be drawn through any angle A of a ſpheric triangle ABC at right angles to the baſe BC, and the point A be not the pole of the baſe; the circumference DAd intercepted by the circle BC is divided in the point A

into two ſegments AD, Ad, of which one as Ad is greater than a quadrant, the other AD is leſs.

For the whole DAd is a ſemicircle, but ſince it is ſuppoſed that A is not the pole of BC, as then every great circle paſſing through A is perpendicular to BC, the ſemicircle DAd is divided unequally in A, and therefore one of the ſegments is greater, the other leſs than a quadrant.

2. When the perpendiculars fall without; the greater perpendicular adjoins to the greater, the leſſer to the leſs ſide of the triangle. For the circle DAd being perpendicular to BC, paſſes through its pole, viz. the pole is in the greater ſegment Ad. Wherefore of all the circumferences of great circles drawn through A to fall on BC, Ad is the greateſt, AD the leaſt; and AC which is nearer to Ad, is greater than AB which is more remote (Cor. Prop. V. B. II.). Therefore,

3. The greater perpendicular Ad is greater than either of the ſides AC, AB, but the leſſer AD is leſs.

4. When the angles at the baſe are of the ſame affection, as one of the perpendiculars falls within the triangle, this is always underſtood to be the perpendicular upon the

baſe, if one be required to be drawn. But if the angles at the baſe be of different affection, then either of the arches AD, Ad will repreſent the perpendicular required, but which is made uſe of will be aſcertained, accordingly as it adjoins to the greater or leſſer ſide.

OF THE PROJECTION OF THE SPHERE.

BOOK IV.

DEFINITIONS.

I. PROJECTION in general is the repreſentation of any object upon a plane, as it would appear to the eye viewing it from any diſtance, and referring every part of the object to its image upon that plane. As all viſion is rectilinear, every point of the object viewed will be ſeen in the viſual ray or right line between the eye and the point, and the repreſentation of the point will be where this right line meets the plane upon which the projection is made; and the interſections of all the viſual rays (proceeding from every point of the object) with the plane form the whole image or projected figure of the object.

II. PRO-

II. PROJECTION OF THE SPHERE is the repreſentation of the ſphere with all its furniture, great and ſmall circles, centres, poles, &c. on the plane of ſome great circle, the eye being ſituated in the axis of the ſphere at right angles to the plane of projection.

III. ORTHOGRAPHIC PROJECTION ſuppoſes the eye to be at a diſtance indefinitely great with reſpect to the diameter of the ſphere itſelf, ſo that the angle which any ray makes with the axis, or the ray paſſing through the centre of the projection, is indefinitely ſmall, and therefore at the ſphere itſelf the rays are aſſumed as parallel to each other, viz. each perpendicular to the plane of projection, becauſe the axis is perpendicular thereto.

THE orthographic repreſentation of any point is therefore where a right line drawn from the point perpendicular to the plane of projection meets the plane; and the orthographic repreſentation of any object is the figure formed by perpendiculars drawn from every point of the object to the plane of projection

IV. THE STEREOGRAPHIC PROJECTION places the eye in the pole of the great circle on which the projection is made.

V. In the ſtereographic projection the pole in which the eye is placed, and where all the rays of viſion meet, or from which all the projecting right lines iſſue, is called the PROJECTING POINT.

VI. The great circle, on whoſe plane the projection is made, is called the PRIMITIVE.

VII. A GREAT circle paſſing through the axis of the primitive, viz. which is perpendicular to the primitive, is called a RIGHT CIRCLE.

VIII. THE LINE OF MEASURES of any circle is the common ſection of the primitive with a great circle perpendicular both to the primitive and that other circle, or it is the right line in which a right circle paſſing through the axis of the other circle cuts the primitive.

IX. A RIGHT line is ſaid to be parallel to a plane, when it is equidiſtant from it, or when perpendiculars drawn from any two points in it to the plane, are equal between themſelves.

X. Two curve lines in the ſame plane, which meet each other are ſaid to TOUCH in the point of their concourſe, when they do not cut each other, viz. when they do not fall partly within, partly without each other. And the contact is ſaid to be EXTERNAL or INTERNAL, as they fall entirely without each other, or the one wholly within the other.

COR. 1. IF a right line be parallel to a plane, it will be parallel to the common ſection which any plane paſſing through it makes with the plane to which it is parallel.

Fig. 30. LET AB be parallel to the plane ED. From any two points therein A, B draw AC, BD perpendicular to the plane ED (11. E. 11.). AC, BD are parallel between themſelves (6. E. 11.), and they are alſo equal (HYP.), therefore AD is a paralelogram (33. E. I.) and AB is parallel to CD, which is the common ſection of the plane ABD with the plane ED.

DRAW AG parallel to CE, then becauſe AB is alſo parallel to CD, the plane GAB is parallel to the plane ECD (15. E. 11.); wherefore any plane cutting GAB in the right line AB will cut the plane ECD in a right line parallel to AB (16. E. 11.).

COR. 2. THE line of meaſures of every circle of the ſphere is a diameter of the primitive.

FOR being the common ſection of the primitive with a great circle, it is a diameter to each of them. (PROP. II. B. II.)

COR. 3. THE line of meaſures of every circle of the ſphere inclined to the primitive is perpendicular to the common ſection of the circle with the primitive.

Fig. 26. 27.

THUS, if ZYz be the primitive, ADB any circle of the ſphere, and ZPz a right circle perpendicular to ADB, whoſe common ſections with the primitive are the right lines XTx, Tz. Becauſe the circle ADB and the primitive are each perpendicular to the right circle ZPz, their common ſection TX is perpendicular to the plane ZPz (19. E. 11.), and therefore to the right line TZ, meeting it in the ſame plane (3. DEF. E. 11.). But TZ is the line of meaſures of the circle ADB (DEF. 8.). Therefore the common ſection of the circle with the primitive, and its line of meaſures are at right angles to each other.

COR. 4. THE line of meaſures of a circle of the ſphere perpendicular to the primitive is a diameter of the primitive, or of the ſphere, at right-angles to the plane of the circle.

FOR the primitive and the right circle making the line of meaſures being each perpendicular to the plane of the circle; their common ſection, which is the line of meaſures, is perpendicular to the plane of the circle.

COR. 5. HENCE the centre and the poles of a circle perpendicular to the primitive are the ſame in the ſphere itſelf as in the projection; and are in the line of meaſures, viz. the centre is the concourſe of the line of meaſures with the common ſection which the circle makes with the primitive, and the poles are the concourſes of the line of meaſures with the circumference of the primitive.

FOR the line of meaſures being perpendicular to the plane of the circle, and being alſo a diameter of the ſphere, its concourſe with the circle, viz. with the common ſection is the centre of the circle (1. and 2. COR. PROP. I. B. II.). And the concourſes of this line of meaſures with the circumference of the primitive, being its extremities on the ſurface of the ſphere, are the poles of the circle (8. DEF. B II.). But the line of meaſures being in the plane of the primitive, every point in it is the ſame in the projection as in the ſphere itſelf.

COR. 6. THE poles of the primitive, and of every circle parallel to the primitive, are projected in the centre of the primitive.

FOR the poles are in a diameter of the ſphere at right-angles to the primitive, wherefore the ſphere being the centre of the primitive alſo, the projected place of the poles is the centre.

COR. 7. IF the projection of a circle of the ſphere meet the primitive, the point in which it meets the primitive will be the ſame with the point of which it is the repreſentative.

IN the orthographic projection of a circle, let E be a point common to the projection and to the circumference of the primitive. To C the centre of the primitive draw EC, alſo EF in the plane of the primitive at right angles to EC, and EH perpendicular to the primitive. Becauſe EH is perpendicular to the primitive, viz. to the plane CEF, it is perpendicular to the right line EC; but EC is alſo perpendicular to EF, wherefore EC is perpendicular to the plane FEH (4. E. 11.). The plane FEH therefore touches the ſphere in the point E (PROP. VII. B. II.), and conſequently EH drawn therein through the point E meets the ſphere in that point only. But becauſe EH is perpendicular to the primitive, the point of which E is the repreſentative is in the right line EH (3. DEF.). wherefore the point repreſented by E is the ſame with E itſelf.

Fig. 22.

IN the ſtereographic projection, let R be the projecting pole, and join RE, RC, EC. The plane RCE will cut the ſphere in a circle, and therefore RE will meet the circumference of the ſphere only in the two points R, E, becauſe otherwiſe it muſt meet the circumference of the circular ſection RCE in more than theſe two points, which is abſurd (2. E. 3.). But the point in the circumference of the ſphere repreſented by E is in the right line RE, and the point R (from the nature of the ſtereographic projection) is incapable of being projected, ſince a right line is not determined by a ſingle point; therefore the point repreſented by E is the ſame with E itſelf.

SCHOL. THESE Corollaries, flowing from the general idea of projection, are delivered as preliminary Propoſitions to both the projections treated of in the two ſubſequent books.

OF THE ORTHOGRAPHIC PROJECTION.

PROP. I. THEOREM.

EVERY right line is projected into a right line, and if the right line to be projected be parallel to the plane of projection, it is projected into an equal right line; but if it meet the plane, the right line will be to its projection as radius is to the coſine of the inclination of the right-line to the plane.

Fig. 26. LET TYC be the plane of projection, and AB any right line without the plane but not perpendicular to it. Draw Aa, Bb perpendicular to the plane TYC (11. E. 11.), then a, b are the projections of the extremities A, B (3. Def.), and are therefore themſelves the extremities of the whole projection. But becauſe Aa, Bb being perpendicular to the ſame plane are parallel between themſelves (6. E. 11.), and AB is a ſtraight line falling upon them, aABb will be a plane (7. E. 11.) perpendicular to the primitive TYC (18. E. 11.), and ab its common ſection with the primitive is a right line (3. E. 11.). But all the right lines drawn from AB at right angles to the plane TYC fall on ab the common ſection of the planes (38. E. 11.), wherefore AB is projected into ab, viz. into a right line.

If AB be parallel to the plane TYC, viz. to the right line ab (1. Cor. Def.), then AaBb is a parallelogram, whoſe oppoſite ſides AB, ab are equal between themſelves, viz. AB is in this caſe projected into a right line equal to itſelf.

But if AB be not parallel to the plane of projection, viz. to ab, let it meet ab in T. Then the angle BTb is the inclination of the right line AB to the plane TYC, (5. Def. E. 11.) and becauſe Aa is parallel to Bb, it will be as AB is to ab, (ſo is BT to Tb, viz.) ſo is radius to the coſine of the angle BTb.

Cor. 1. A plane angle parallel to the primitive is projected into an equal angle.

FOR each of the right lines containing the plane angle is projected into a parallel right line, and therefore the angle comprehended by the projected right lines will be equal to the plane angle of which it is the projection (10. E. 11.).

COR. 2. HENCE any plane rectilinear figure parallel to the primitive is projected into an equal ſimilar right-lined figure.

COR. 3. A RIGHT line perpendicular to the plane of projection, is projected into a point.

COR. 4. EVERY right line in a plane parallel to the primitive is projected into an equal and parallel right line.

PROP. II. THEOREM.

EVERY circle of the ſphere perpendicular to the primitive, is projected into a diameter of the circle, at right angles to its line of meaſures.

LET YBV be a circle perpendicular to TYC the plane of projection, their common ſection VY is a diameter to the circle YBV(II. or III. B. II.), and I ſay the circle YBV is projected into YV. For the perpendiculars drawn from every point of the circle YBV to the plane TYC fall on the common ſection YV (11. E. 11.), and between the extremities Y,V, becauſe the perpendiculars at theſe extremities touch the circle (Cor. 16. E. 3.), and therefore include the whole circle within them. Wherefore the circle YBV is projected into its diameter YV. And becauſe this diameter is the common ſection of the circle with the primitive, it is perpendicular to the line of meaſures of the circle (3. Cor. Def.).

Cor. If the circle were elevated above the primitive, but its plane if continued were perpendicular to the primitive, yet by the ſame reaſoning would its projection be ſhewn to be a right line equal to its diameter. And alſo, that any plane figure perpendicular to the primitive will be projected into a right line.

Cor. 2. All right circles are projected into right lines, which are diameters of the primitive.

PROP. III. THEOREM.

A CIRCLE parallel to the primitive, is projected into an equal circle, whoſe centre is the ſame with the centre of the primitive.

Fig. 22. LET BTQ be a circle of the ſphere parallel to EGa the primitive; and *btq* its projection; I ſay *btq* is a circle equal to BTQ, and parallel to EGa.

LET *b*, *t* be any two points in the projection *btq*, and B, T the correſpondent points in the circle BTQ, and C, O being the centres of the primitive, and of the circle BTQ, join Co, C*b*, C*t*, OB, OT, B*b*, T*t*. Becauſe the circles BTQ, EGa are parallel, the right line OC joining their centres is perpendicular to each of them (4. COR. PROP. I. B. II.), wherefore the centre O is projected into the centre C of the primitive (3. DEF.), and the ſemidiameters BO, TO are projected into *b*C, *t*C. But becauſe the planes BTQ, *btq* are parallel, *b*C, *t*C are epual to BO, TO, (4. COR. PROP. I.), viz. becauſe BO, TO are equal between themſelves, *b*C, *t*C are alſo equal between themſelves. By the ſame reaſoning is it ſhewn that right lines drawn from every point of the projection *btq* to the centre C of the primitive are equal between themſelves, and each equal to the ſemidiameter of the circle BTQ. Therefore *btq* is a circle concentric with the primitive, and equal to that from which it is projected.

Cor. The radius of the projection is equal to the coſine of the circle's diſtance from the primitive, or to the ſine of its diſtance from the pole of the primitive, for the radius of every leſſer circle of the ſphere parallel to the primitive is the ſaid coſine or ſine to the radius of the ſphere.

PROP. IV. THEOREM.

Every circle of the ſphere inclined to the primitive is projected into an Ellipse, whoſe centre is the projection of the centre of the circle; and any two conjugate diameters of the inclined circle are projected into two conjugate diameters of the ellipſe.

Note. Any two diameters of a circle at right angles to each other are called conjugate diameters of that circle.

Let

Fig. 26. n. 1. 2. and 27. 28. Let ADB be any circle of the ſphere inclined to the primitive ZYC, I ſay its projection adb on the plane of the primitive is an ellipſe, and any two conjugate diameters of the circle will be projected into two conjugate diameters of the ellipſe.

Let O be the centre of the inclined circle, DOE, FOG. two conjugate diameters of it, and doe, fog the projections of DOE, FOG. Join Dd, Ee, Ff, Gg which will be perpendicular to the plane of the primitive (3. Def.), and the planes DdeE, FfgG will therefore each be perpendicular to the primitive (18. E. 11.). Wherefore Oo their common ſection is perpendicular to the primitive (19. E. 11.), and Dd, Ee, Ff, Gg, Oo are all parallel (6. E. 11.). But becauſe in the plane DdeE the right lines DE, de fall upon the three parallels Dd, Oo, Ee, they will be proportionally divided by them (Lem. IV.), therefore DE being biſected in O, de is biſected in o. For the ſame reaſon is fg biſected in o, and every right line drawn through o and terminated by the ſection is biſected in o. Wherefore o which is the projection of the centre O of the circle is the centre of the whole projection adb; and de, fg are diameters of the projection.

Through any point k in either of theſe diameters as fg draw hki parallel to the other de, and meeting the line bounding the projection in h, i. Let H, I be the points in the circumference ADB from which h, i are projected, and join Hh, Ii, HI. As before it is ſhewn that HhiI is a plane at right angles to the primitive, and that Kk, which is the common ſection of the planes HhiI, FfgG, divides the right lines hi, HI, and fg, FG proportionally in k, K. But becauſe the ſtraight lines de, hi are parallel, and alſo Dd, Hh being perpendicular to the primitive are parallel between themſelves (6. E. 11.), the planes DdeE, HhiI will be parallel (15. E. 11.), wherefore alſo HI, DE the common ſections of the ſame planes with the inclined circle will be parallel (16, E. 11.). But DE is perpendicular to FG, therefore HI is alſo perpendicular to FG, and conſequently is biſected thereby in the point K (3. E. 3.), and it has been ſhewn that hi is divided in the ſame proportion in k, therefore the diameter fg of the projection biſects the right line hi terminated by the projection and drawn parallel to the other diameter de. By the ſame reaſoning will it be ſhewn that the diameter fg biſects all the parallels to the diameter de, therefore de, fg are conjugate diameters (17. Def. Apol. Lib. I.).

Lastly, becauſe FG, fg are proportionally divided in O, o, and K, k, and in four proportionals, ſimilar right-lined figures deſcribed upon the antecedents, are proportional to any two ſimilar right-lined figures deſcribed upon the conſequents (22. E. 6.) the ſquare of FO will be to the ſquare of fo as the ſquare of FK is to the ſquare of fk, and as the ſquare of FK is to the ſquare of fk ſo is the rectangle FKG to the ſimilar rectangle fkg. Therefore ex æquo and by alternation, the ſquare of FO will be to the rectangle FKG as the ſquare of fo is to the rectangle fkg. But from the property of the circle, FO is equal to DO, and the rectangle FKG to the ſquare of HK, therefore the ſquare of DO is to the ſquare of HK as the ſquare of fo is to the rectangle fkg. But becauſe in the plane ADB are drawn DO, HK parallel to each other, and Dd, Oo, Hh, Kk perpendicular to another plane ZYC, the ſegments intercepted between the perpendiculars in one are proportional to the ſegments intercepted between them in the other (6. LEM.), wherefore DO is to HK as do to hk, and the ſquare of DO to the ſquare of HK as the ſquare of do to the ſquare of hk. Therefore ex æquo, the ſquare of fo is to the rectangle fkg as to the ſquare of do is to the ſquare of hk, and by alternation, the ſquare of fo is to the ſquare of do as the rectangle fkg is to the ſquare of hk, which is the diſtinguiſhing property of the ellipſe.

COR. 1. HENCE in an elliptical projection, if a ſemicircle be deſcribed on any diameter of the projection, and from a point in the diameter be drawn two right lines, the one meeting the ellipſe and parallel to the conjugate diameter, the other at right angles to the diameter of the circle and meeting the circumference, the right line meeting the circle will be to the right line meeting the ellipſe as the ſemidiameter of the circle is to the ſemiconjugate diameter. Let AB, DC be two conjugate diameters of the elliptical projection ADB, and round the centre O with the diſtance AO or OB let a circle be deſcribed, and at any point E in the diameter AB let EF be drawn parallel to DO meeting the ellipſe in F, and EI at right angles to AB meeting the circumference of the circle in I; I ſay, as IE is to FE ſo is AO to DO. *Fig.* 49.

For the rectangle AEB is to the ſquare of FE as the ſquare of AO is the ſquare of DO. But the rectangle AEB is equal to the ſquare of IE (Lem.), therefore the ſquare of IE is to the ſquare of FE as the ſquare of AO is to the ſquare of DO, and IE is to FE as AO to DO.

Cor. 2. Hence if HG, LG be drawn in like manner from any other point G in the diameter AB, it will be as LG is to HG, ſo is IE to FE.

For LG is alſo to HG as AO to DO, therefore ex æquo, LG is to HG as IE to FE.

Cor. 3. If from any number of points O, E, G in the diameter AB be drawn to the ellipſe OD, EF, GH parallel to the ordinates applied to the diameter AB, and OK, EI, GL to the circle at right angles to AB, and towards the ſame parts of the diameter, KD, IF, LH being joined will be all parallel between themſelves.

For becauſe of the parallels, the angles KOD, IEF, LGH will be equal between themſelves, and the ſides about theſe angles are proportional, therefore the homologous angles OKD, EIF, GLH are equal to each other (4 E. 6.). But KO, IE, LG are parallel to each other, therefore KD, IF, LH are alſo parallel.

Fig. 26. n. 1. Cor. 4. If the circle *adb* of the ſphere on the oppoſite ſide of the primitive, equal to the circle ADB have the ſame inclination to the primitive, but in a contrary poſition, ſo as to have the ſame common ſection with the primitive that ADB has, then will it be repreſented by the ſame projection a d b.

And in like manner every leſſer circle parallel to the primitive will have the ſame projection with an equal parallel circle on the oppoſite ſide of the primitive.

SCHOLIUM. FROM this projection all the capital properties of the ellipſe are eaſily deduced, as all right lines belonging to the circle, touching or cutting it, or dividing each other in any proportion are projected into ſimilar right lines touch or cutting the ellipſe, which ſingle conſideration almoſt immediately transfers the chief properties of the circle to the like properties in the ellipſe.

THIS projection is the ſection of a ſcalene cylindre, whoſe baſe is the circle to be projected, and ſuperficies the whole collection of parallel rays falling on every point of the circumference. And in like manner as in the ſcalene cone, the ſubcontrary ſection of it, which is the other circle taken notice of in COR. 4., is a circle, but every other ſection of the cylindre is an ellipſe.

THE three firſt Corollaries are alone of all the properties of the ellipſe annexed to this Propoſition, becauſe they alone are neceſſary to the ſubſequent problems, by which means thoſe who have had no converſation with the conic ſections, have here every thing demonſtrated of the ellipſe which is neceſſary to the orthographic projection.

PROP. V. THEOREM.

IN the elliptical projection of every oblique circle of the ſphere, the major axis is equal to the diameter of the circle, and the minor axis is ſituated in the line of meaſures, and is in proportion to the major axis as the coſine of the angle of inclination of the circle to the primitive is to the radius.

Fig. 26. n. 1. 2. and 27. 28. THE ſame things remaining, let ZAP be the right circle paſſing through the pole P of the oblique circle ADB, cutting the ſame in the diameter AB (III. PROP. B. II.), and the primitive in the line of meaſures Zz. Becauſe the circle ZAP is penpendicular to the primitive, the perpendiculars drawn from A, B to the primitive fall on the common ſection Zz (38. E 11.), wherefore aob the projected diameter of AOB is in the line of meaſures Zz. Draw the diameter ROS at right angles to AB, and let ros be the projection of ROS. Then, by the preceding, ab, rs are conjugate diameters of the ellipſe adb into which the circle ADB is projected. But becauſe the plane ADB is perpendicular to the plane ZPz, and RS is drawn in the former at right angles to the common ſection AB, therefore

RS will be perpendicular to the plane ZAP (4. DEF. E. 11.); and conſequently the plane RSsr, which paſſes through RS will be perpendicular to ZAP. But the primitive is alſo perpendicular to ZAP, therefore rs, which is the common ſection of the primitive with the plane RSsr, is perpendicular to ZAP (19. E. 11.). Wherefore rs is at right angles to aob (3. DEF. E. 11.), viz. the conjugate diameters ros, aob are the axes of the ellipſe. And becauſe rs, RS have been ſhewn to be each perpendicular to the ſame plane ZAP, they are parallel between themſelves (6. E. 11.), and conſequently rs is equal to RS (PROP. I.), a diameter of the circle ADB. Alſo the plane ZAP being perpendicular to each of the planes ADB, ZYC, the inclination of the right line AB to the right line Zz meaſures the inclination of the planes ADB, ZYC to each other (2. COR. PROP. VI. B. II.). But this is alſo the ſame with the inclination of the right line AB to the plane ZYC. Wherefore AB or rs is to ab as radius to the coſine of the inclination of the circle ADB to the primitive (PROP. I.).

COR. 1. THE vertices of the axis minor of every leſſer elliptical projection are in the line of meaſures, at diſtances from the center of the primitive equal to the ſines of the circle's diſtances from the neareſt pole of the primitive.

FOR N being the neareſt pole of the primitive in the right circle ZAP, and C the centre of the primitive in the line of meaſures Zz, the diſtances of the circle from the pole N are meaſured by the arches AN, BN, of which arches Ca, Cb are the ſines.

COR. 2. THE axis minor of every leſſer elliptical projection is equal to the ſum or difference of the ſines of the circle's diſtances from the pole of the primitive, accordingly as the circle encompaſſes the pole of the primitive, or has the pole without it; viz. as the diſtance of the circle from its own pole is greater or leſs than the diſtance of its pole from the pole of the primitive.

COR. 3. THE centre of every leſſer elliptical projection is in the line of meaſures, and its diſtance from the centre of the primitive is to the coſine of the circle's diſtance from its own pole as the ſine of the circle's inclination to the primitive is to radius.

JOIN COP, which is in the plane of the right circle ZAP, and being perpendicular to the plane ADB (2 COR. PROP. I. B. II.), it makes right angles with AB, wherefore AO is the ſine, and CO the coſine of the arch AP, and in the right-angled triangle COo, Co is to CO the coſine of AP, as the coſine of the angle OCo, viz. the ſine of the inclination CTO, is to radius.

COR. 4. THE centre of the projection of every great circle is the centre of the primitive itſelf.

FOR the centre of every great circle, being the centre of the ſphere, viz. of the primitive, is the ſame in the projection as in the circle itſelf, and this projected centre is the centre of the ellipſe (PROP. IV.).

COR. 5. THE axis major of the elliptical projection of every great circle is the common ſection of the great circle with the primitive, and is equal to the diameter of the ſphere. And the axis minor is double to the coſine of the great circle's inclination to the primitive.

FOR the primitive being alſo a great circle, their common ſection is a common diameter to each (PROP. II. B. II.), wherefore the common ſection paſſes through the centre of the primitive, viz. the centre of the elliptical projection, by the preceding Corollary. The common ſection is therefore a diameter of the ellipſe, and it is perpendicular to the line of meaſures (3. COR. DEF.), viz. to the minor axis, wherefore it is the major axis itſelf. The latter part of the Corollary is included in the Propoſition itſelf.

COR. 6. THE projected pole of every inclined circle is in the line of meaſures at a diſtance from the centre of the primitive equal to the ſine of the diſtance of the circle's pole from the pole of the primitive, or of half the ſum or half the difference of the circle's greateſt and leaſt diſtances from the pole of the primitive (accordingly as the circle encompaſſes the pole of the primitive, or lies wholly without it), or to the ſine of the circle's inclination to the primitive; for theſe are one and the ſame (2. COR. PROP. V. B. II.).

FOR Pp being drawn perpendicular to the line of meaſures, viz. to the primitive, p is the projection of P, and Cp is the ſine of the arch PN.

PROP. VI. THEOREM.

IN the elliptical projection of every oblique circle of the ſphere, the ſemiaxis major or the ſemidiameter of the circle is to the excentricity of the ellipſe (viz. the diſtance of the centre from the focus), as radius is to the ſine of the inclination of the circle to the primitive. And the ſemiaxis minor is to the excentricity as radius is to the tangent of the inclination.

Case 1. When the projection is of a great circle. *Fig.* 28.

Let ARBS be a great circle inclined to the primitive in the angle ACZ, and aRbS the elliptical projection on the plane of the primitive ZRzS, and every thing remaining as in the preceding, aCb, RCS being the axes; let U be one of the foci in the axis major RS; I ſay, RC is to CU as radius is to the ſine of the angle ACZ, and aC is to CU as radius is to the tangent of the angle ACZ.

Join aU. Becauſe from the property of an ellipſe, aU is equal to CR, viz, to AC, therefore in the triangles aCU, CaA, the ſides aC, aU are equal to the ſides aC, CA, and of the correſpondent angles, the two aCU, CaA are equal being each right, while the remaining two aUC, CAa are each leſs than right, wherefore the remaining ſides CU, Aa are equal between themſelves, as alſo the remaining angles CaU, aCA, (Lem. I.). Wherefore aU or RC is to CU (as AC is to Aa, viz.), as radius is to the ſine of the angle ACa or ACZ, and aC is to CU (as aC is to Aa, viz.) as radius is to the tangent of the angle ACZ.

Case 2. When the projection is of a leſs circle.

Every thing remaining the ſame as in the preceding propoſition, let U be one of the foci in the axis major rs; I ſay, 26. 27.

ro is to oU as radius to the ſine of the angle OTC the inclination of the circle ARS to the primitive ZYz, and ao is to oU as radius to the tangent of the angle OTC.

JOIN aU, and draw AW parallel to ao meeting Oo in W. Becauſe from the property of an ellipſe, aU is equal to ro, viz. to AO (PROP. V.), and ao is equal to AW, therefore aU, ao, are equal to AO, AW, each to each, alſo the correſpondent angles aoU, AWO are equal being right, and the other correſpondent angles aUo, AOW are each leſs than a right angle, wherefore the remaining ſides oU, OW, and the remaining angles oaU, OAW are equal between themſelves (LEM. I.). Therefore aU or ro is to ou (as AO is to OW, viz. as TO is to Oo, viz.) as radius is to the ſine of the angle OTC. And ao is to oU as AW to OW, viz. (as TO is to Oo, viz.) as radius is to the tangent of the angle OTC.

COR. 1. IN the elliptical projection of every great circle, the excentricity is equal to the ſine of its inclination to the primitive.

Fig. 28. FOR CU is equal to Aa, the ſine of the angle ACZ.

COR. 2. IF in any elliptical projection the focus and either vertex of the minor axis be joined, the angle comprehended by the right line drawn and the minor axis is equal to the inclination of the projected circle.

For it was ſhewn that the angle CaU is equal to the angle ACa, when the projection is of a great circle; as alſo that the angle oaU is equal to the angle OAW, viz. to the angle OTC. *Fig.* 28. 26. 27.

PROP. VII. THEOREM.

If de, fg be two conjugate diameters of an elliptical projection, cutting each other in the centre O, and ik parallel to the one de meet the other fg in k, and the ellipſe in i, on fg let a ſemicircle fQg be deſcribed, and oQ, kM be drawn at right angles to fg and meet the circumference of the circle in Q, M; I ſay the arch QM ſhall be ſimilar to the arch of which ei is the projection. 27.

Every thing remaining as in Prop. V., join OI, oM. Becauſe it is there demonſtrated that OF, of are proportionally divided in K, k, therefore OF is to OK as of is to ok; but OF is equal to OI, and of to oM, therefore OI is to OK as oM to ok. Wherefore two ſides of the triangles OIK, oMk being proportional, and of the homologous angles the two OKI, okM being equal, becauſe each a right one, and therefore the other two OIK, oMk each leſs than a right one, the angles OIK, oMk will alſo be equal between themſelves (7. E. 6.). Wherefore the alternate angles IOE, MoQ will alſo be equal. But in circles, equal angles, at the centres, are ſubtended by ſimilar circumferences (LEM. VIII.), therefore the circumference QM is ſimilar to the circumference EI, viz. to the circumference of which ei is the projection.

Fig. 49. COR I. IF AOB, DOC be two conjugate diameters of an elliptical projection, O the centre, and on either of them as AB a ſemicircle be deſcribed, from two points E, G in which are drawn to the ellipſe EF, GH parallel to DO, and to the circumference of the circle EI, GL at right angles to AB, the arch IL ſhall be ſimilar to the arch from which FH is projected.

DRAW OK perpendicular to AB, then the arches IK, KL are reſpectively ſimilar to the arches of which FD, HD are the repreſentatives, therefore the whole or remaining arch IL (accordingly as the points I, H are towards different or the ſame parts of D) is ſimilar to the whole or remaining arch repreſented by FH.

Cor. 2. If the conjugate diameters be the axes, then the parallels to the conjugate and the perpendiculars to the diameter of the circle coincide, and if the ſemicircle be deſcribed on the major axis, becauſe the major axis is equal to the diameter of the circle from which the projection is made (Prop. V.), the intercepted arch of the ſemicircle will be equal to the arch of which the intercepted arch of the ellipſe is the repreſentative.

For ſimilar arches of equal circles are themſelves equal (36. E. 3.).

PROP. VIII. THEOREM.

THE projection of every circle of the ſphere is wholly comprehended within the primitive.

Because the perpendiculars to the plane of the primitive at every point of the circumference meet the ſphere only in the circumference of the primitive (as is demonſtrated in the 7. Cor. Def.), therefore the whole collection of theſe perpendicular rays form the ſuperficies of a right cylindre whoſe baſe is the primitive, and the ſphere is wholly comprehended within this cylindre. But becauſe all perpendiculars to the ſame plane are parallel between themſelves (6. E. 11.), therefore no perpendicular drawn from any point of the ſphere to the primitive can fall without the primitive, viz. its concourſe with the primitive cannot be without the circumference of the primitive. But by the concourſe of ſuch perpendiculars with the plane of the primitive is every projection determined (3. DEF.), therefore the projection of a circle of the ſphere cannot in any point fall without the circumference of the primitive, viz. it is wholly comprehended within the limits of the primitive.

COR. IF on the ſurface of the ſphere the circumference of a leſs circle fall wholly without that of the primitive, its projection will in every point fall within the circumference of the primitive. And converſely, if a projection fall in every point within the circumference of the primitive, it will repreſent a leſs circle, which on the ſurface of the ſphere falls wholly without the circumference of the primitive.

PROP. IX. THEOREM.

IF on the ſurface of the ſphere a leſs circle meet the circumference of the primitive in one point only, viz. touch the primitive, its projection will touch the primitive internally in the ſame point, and the point of contact will be one of the vertices of the axis minor of the projection. And converſely, if a projection meet the primitive in one point only, it touches it in that point, the point of contact will be a vertex of the minor axis, and the circle repreſented by it will be a leſs circle of the ſphere touching the primitive in the ſame point.

For if a leſs circle of the ſphere touch the primitive on the ſurface of the ſphere, its projection meets the primitive in the ſame point, and in that point only, becauſe in whatever other point it were ſuppoſed to meet the primitive, in the ſame point would the circle repreſented by it meet the primitive again on the ſurface of the ſphere (7. Cor. Def.), which is abſurd. Therefore the projection meets the primitive in one point only, and being wholly comprehended within the primitive, it touches it internally in that point (10. Def.). Moreover, becauſe the leſs circle touches the primitive on the ſurface of the ſphere, the right circle paſſing through the point of contact will paſs through the poles of the leſs circle alſo (XII. Prop. B. II.). Wherefore this right circle cuts the primitive in the line of meaſures (8. DEF.), and conſequently the point of contact will be one extremity or vertex of the minor axis (PROP. V.).

AND converſely, if a projection meet the primitive in one point only, becauſe every projection is wholly comprehended within the primitive, it therefore touches the primitive internally in the point of concourſe. Wherefore the circle repreſented by the projection meets the primitive on the ſurface of the ſphere in the ſame point, and in that point only, becauſe if it meet the primitive in more than that point, then in another point would the projection alſo meet the circumference of the primitive (7. COR. DEF.), which is contrary to the hypotheſis. Therefore the circle repreſented by the projection touches the primitive on the ſurface of the ſphere (12. DEF. B. II.), and conſequently is a leſs circle, becauſe a great circle would cut the primitive, which is alſo a great circle, in two points the extremities of their common diameter (PROP. II. B. II.). Wherefore, the circle repreſented by the projection touching the primitive on the ſurface of the ſphere, for the ſame reaſon as in the preceding part of this PROP., will the line of meaſures paſs through the point of contact, and conſequently the point of contact be a vertex of the minor axis of the projection.

PROP.

PROP. X. THEOREM.

IF a circle of the ſphere meet the primitive in two points on the ſphere's ſurface, in the ſame two points will its projection touch the primitive. And Converſely, If the projection of a circle of the ſphere meet the primitive in two points, it will touch the primitive in each of theſe points, and will be the projection of a circle cutting the primitive in the ſame two points on the ſurface of the ſphere.

For in the ſame two points on the ſurface of the ſphere in which the circle meets the primitive, will its projection meet the primitive (7. Def.); but the projection is wholly comprehended within the circumference of the primitive (Prop. VIII.); therefore the projection touches the primitive in each point of concourſe (10. Def.).

And Converſely, If the projection of a circle of the ſphere meet the primitive in two points, becauſe in no part it falls without the circumference of the primitive (Prop. VIII.), it therefore touches the primitive in each point of concourſe.

Alſo in the ſame points in which it meets the primitive, does the circle repreſented by it meet the circumference of the primitive on the ſurface of the ſphere.

COR. 1. THE right line joining the two points of contact is perpendicular to the line of meaſures of the circle.

FOR this right line is the common ſection of the primitive with the circle, therefore it is perpendicular to the line of meaſures (3. COR. DEF.).

COR. 2. THE elliptical projection of every great circle touches the primitive in two points, which are the vertices of its major axis.

FOR every great circle inclined to the primitive cuts it on the ſurface of the ſphere in two points, which are the extremities of its common ſection with the primitive (PROP. II. B. II.). But in theſe points the projection touches the primitive, and this common ſection is the major axis of the elliptical projection (5. COR. PROP. V.). Therefore, &c.

PROP. XI. THEOREM.

IF to the line of meaſures of any projected circle, at the vertices of the diameter of the projection ſituated in the line of meaſures, perpendiculars be drawn (towards the ſame or contrary parts, accordingly as the projection does not meet the primitive at all, or touches it); the two points in which theſe perpendiculars meet the circumference of the primitive, and the point in which the line of meaſures meets the common ſection of the circle with the primitive, ſhall be in one ſtraight line, and the angle which this ſtraight line makes with the line of meaſures ſhall be equal to the inclination of the projected circle to the primitive.

Case 1. When the projected circle is parallel to the primitive.

Every thing remaining the ſame as in Prop. VI., draw aL, bY perpendicular to ab (towards the ſame or contrary parts of ab, accordingly as the projection adb repreſents a circle wholly without the primitive, or meeting it on the ſurface of the ſphere); I ſay, T, L, Y are in one ſtraight line, and that the angle which this ſtraight line makes with

Fig. 26. n. 1. 2. and 27. 28.

TC meaſures the inclination of the circle ADB to the primitive.

JOIN TY, TL, and CY, CB. Becauſe in the triangles CbY, CbB, the ſide CY is equal to the ſide CB, and Cb is common, and of the homologous angles, the two CbY, CbB are equal, being each a right one, and conſequently the other two CYb, CBb each leſs than a right angle, therefore the baſes bY, bB are equal between themſelves (LEM. I.). For the ſame reaſon, is aL equal to aA. But becauſe of the parallels aA, bB, it will be as Ta is to Tb, ſo is aA to Bb, therefore Ta will be to Tb as aL to bY, and Ta, Tb are in one ſtraight line, alſo aL, bY are parallel, wherefore TL, TY are in one ſtraight line (32. E. 6.). But becauſe Tb, bY are equal to Tb, bB, each to each, and the angles TbY, TbB are equal, being each right, the correſpondent angles bTY, bTB are equal between themſelves (4. E. 1.). But the angle bTB meaſures the inclination of the plane, viz. of the circle ADB, to the primitive ZYz, therefore the angle bTY is equal to the inclination which the circle repreſented by the projection adb has to the primitive.

COR. 1. THE right line LY is parallel to the right line aU drawn from the vertex a of the minor axis to one of the foci.

FOR the angle CaU is alſo equal to the inclination (2. COR. PROP. VI.), therefore the alternate angles CTY, CaU, being equal to one and the ſame magnitude, are equal between themſelves, and conſequently LY is parallel to aU.

COR. 2. LY is equal to the diameter of the circle repreſented by the projection adb, and the arch L*p*Y is double to the diſtance of the circle from its pole.

FOR by the ſame reaſoning as above, is TY ſhewn to be equal to TB, and TL to TA, therefore the whole or remainder LY is equal to the whole or remainder AB, viz. to the diameter of the circle ADB. But becauſe equal right lines in equal circles intercept equal circumferences (28. E. 3.), the circumference L*p*Y (is equal to the circumference APB, viz.) is double to the circumference AP (2. COR. PROP. IV. B. II.), which is the diſtance of the circle ADB from its pole P.

COR. 3. HENCE if the line of meaſures of an elliptical projection be given, and the right line LY in poſition and magnitude, every thing relating to the circle repreſented by the projection, is projected from the primitive as from the right circle APB.

FOR C*op* being drawn perpendicular to LY, and meeting the primitive in *p*, join Yb, La, *oo*, *p*p. Becauſe the angles CT*o*, CTO are each equal to the inclination, and the angles C*o*T, COT are each right, the triangles C*o*T, COT are equiangular, and CT being a common ſide and oppoſite to equal angles, the ſides C*o*, CO, and T*o*, TO are equal (26. E. 1.). Wherefore two ſides in the triangles T*o*o, TOo being equal, each to each, and the contained angles *o*To, OTo, the correſpondent angles *o*oT, OoT will be equal. Conſequently, *o* o is perpendicular to the line of meaſures CT, and therefore the point *o* being given, the centre o of the projection is given. By the like reaſoning, are *p*p, Yb, La ſhewn to be perpendicular to CT, and therefore the points *p*, Y, L being given, the projected pole p, and the vertices b, a of the minor axis ab are given.

ALSO, If at the centre C of the primitive be drawn Cn perpendicular to the line of meaſures CT, and meet the primitive in n, the arch *p*n will in like manner be ſhewn to be equal to the arch PN, the diſtance of the pole of the circle ADB from the nearer pole N of the primitive, or the meaſure of the circle's inclination to the primitive (PROP. VI. B. II.); and the arches Yn, Ln to be equal to the arches BN, AN the leaſt and greateſt diſtances of the circle from the ſame pole N. Therefore the line of meaſures CT, and the points *p*, Y, L in the circumference of the primitive being given, the arches *p*n, Yn, Ln equal to the inclination of the circle ADB to the primitive, and to its leaſt and greateſt diſtances from the nearer pole of the primitive are given.

LEM. IX. PROBLEM.

AN ellipſe being given in poſition and magnitude, to find the centre of it.

LET AEF be an ellipſe given in poſition and magnitude, whoſe centre is required. Through any two points within the ellipſe draw two right lines Hh, Ff parallel to each other, and terminated by the ellipſe; biſect Hh, Ff in G, E, through which points draw the right line EG meeting the ellipſe in A, B. The point O in which AB is biſected ſhall be the centre of the ellipſe. *Fig.* 49.

FOR AB is a diameter, and conjugate to that to which Hh, Ff are parallel, becauſe the diameter which is conjugate thereto does biſect Hh, Ff (PROP. IV.); therefore if AB be not this conjugate diameter, then there is another right line which will alſo biſect Hh, Ff; viz. two different right lines may be drawn through the ſame two points, which is abſurd. Wherefore AB is a diameter of the ellipſe, and becauſe every diameter is biſected in the centre, the point O is the centre, which was required.

Cor. 1. Hence the axes of a given ellipſe are eaſily found.

Fig. 45. For the centre C of an ellipſe ARB being found, deſcribe a circle round this centre, and paſſing through any point E of the ellipſe. If this circle fall wholly within, or wholly without the ellipſe, the right line joining C, E will be accordingly the minor or major axis of the ellipſe. But if the circle cut the ellipſe, let e be one of the points in which it cuts it towards the ſame parts of C, and joint Ee. The diameter ACB parallel to Ee ſhall be one of the axes, and RCS perpendicular to Ee ſhall be the other. Let RS meet Ee in Q, in which point it will biſect Ee (3. E. 3.), therefore Ee is ordinately applied to the diameter RS, and conſequently the diameter AB which is parallel to Ee will be conjugate to RS. But becauſe RS is perpendicular to Ee, it will be alſo perpendicular to AB, and therefore RS, AB being conjugate diameters and at right angles to each other, they are the axes of the ellipſe.

Cor. 2. The axes of an ellipſe being found, the foci are given.

Fig. 44. For AB being the minor, and RS the major axis of an ellipſe ARB, round the centre A of the minor axis with the diſtance of the ſemiaxis major deſcribe a circle meeting the axis major in U, u. Theſe ſhall be the foci of the ellipſe.

LEM. X. PROBLEM.

TO deſcribe an ellipſe, whoſe axes are given in poſition and magnitude.

Let AB, RS be two right lines, biſecting each other in C; it is required to deſcribe the ellipſe, whoſe axes are AB, RS. *Fig.* 52.

Round the centre C, with the diſtance of either ſemiaxis CR deſcribe a circle RGSg, and let a right line FDf continually perpendicular to RS move through the whole axis RS, the points D, d, in wich the ſegments FH, fH, intercepted between the circle and the axis, are divided in the ratio of RC to AC, will by their motion deſcribe the ellipſe required. The demonſtration immediately depends upon the 1. Cor. Prop. IV.

OTHERWISE.

Round the vertex A of the minor axis AB, and with a diſtance equal to the ſemiaxis major CR deſcribe a circle meeting RS in U, u. Let a thread UIu equal in length to the axis major RS, and fixed at its two extremities in the points U, u, be kept continually ſtretched by a ſtile as at I. If the ſtile I, thus ſtretching the thread, be carried round till it has performed an entire revolution, it will by its motion deſcribe an ellipſe, whoſe axis major is RS equal to the length of the thread, and minor axis is AB.

OTHERWISE.

LET a right line KIL equal in length to the ſemiaxis major and minor together, viz. the part KI equal to RC, and the remaining part IL equal to AC, be moved; ſo as the extremity K always be carried in the axis minor, and the extremity L in the axis major. Then will the point I by this motion, when it has performed an entire revolution, deſcribe an ellipſe whoſe axes are RS, AB.

THE demonſtration of the two latter methods is more proper to a treatiſe of conic ſections; but it will not perhaps be unacceptable to have annexed to the firſt ſolution of this Problem theſe two latter very elegant methods of deſcribing an ellipſe whoſe axes are given.

THE firſt method is equally applicable to any two conjugate diameters, which are given in poſition and magnitude, and the third, which is by far the moſt convenient for practice, may alſo eaſily be adapted to the ſame data.

By any one of them a number of points through which the ellipſe paſſes, may be found, and therefore the whole curve eaſily traced by the point of a pen.

PROP. XII. PROBLEM.

THE primitive, and the projection of a circle inclined thereto being given, to draw its line of meaſures.

Case 1. When the projection is that of a great circle perpendicular to the primitive.

Because the projection is a right line, let RS be the rectilinear projection of a circle perpendicular to the primitive. GCg a diameter of the primitive at right angles to RS, ſhall be the line of meaſures of the circle repreſented by RS (Prop. II.). *Fig.* 47. 48.

Case 2. When the projection is of a circle inclined obliquely to the primitive.

Fig. 46. IF it be the projection of a great one, as RASB, it will touch the primitive in two points R, S (2. COR. PROP. X.), and the right line RS joining theſe points is perpendicular to the line of meaſures (1. COR. PROP. X.). Wherefore the diameter ACB perpendicular to RS ſhall be the line of meaſures required.

43. BUT if the projection be of a leſs circle, as ARBS, find O the centre of the elliptical projection (LEM. IX.), through which and C the centre of the ~~projection~~ primitive draw the right line GOg. This ſhall be the line of meaſures required. For the line of meaſures paſſes through the centre of the primitive (2. COR. DEF.), and the centre of the ellipſe (3. COR. (PROP. V.), therefore GOg ſhall be the line of meaſures ſought.

PROP. XIII. PROBLEM.

THE projection of any circle of the ſphere being given, to find the projection of its pole.

CASE 1. IF the projected circle be parallel to the primitive, the projection of its pole will be in the centre of the primitive (6. COR DEF.).

Case 2. If the circle be perpendicular to the primitive, then the poles of that circle will be in the circumference of the primitive, be the ſame with the projected poles, and will be found in the interſections of the line of meaſures with the circumference of the primitive (5. Cor. Def.). Wherefore a diameter of the primitive at right angles to the right line repreſenting the projected circle, will cut the primitive in two points, which are its poles.

Case 3. When the projected circle is inclined to the primitive.

Fig. 43. 44.

Let ARB be the elliptical projection of any oblique circle, through the centre of which and C the centre of the primitive draw the line of meaſures CBA, meeting the ellipſe in B, A, and the primitive in G, g. Draw CH, BK, AT perpendicular to Gg, meeting the primitive in H, K, T. Biſect the arch KT in L, and draw LP perpendicular to Gg; P ſhall be the projected pole of the circle, of which ARB is the elliptical projection (3. Cor. Prop. XI.).

Otherwise.

From either vertex A of the minor axis to either focus U draw AU, and CL perpendicular to AU meeting the primitive in L; draw LP at right angles to CG, and P ſhall be the projected pole.

Let CL meet AU in I, becauſe CI is perpendicular to AI, the angles CAI, ACI are equal to a right angle, viz. to the angle ACH, wherefore taking away the common angle ACI, the remaining angle CAI ſhall be equal to the remaining angle LCH. But the angle CAI is equal to the inclination of the circle to the primitive (2. Cor. Prop. VI.), therefore the angle LCH is equal to this inclination, and PC is the ſine thereof. Wherefore P is the projected pole (6. Cor. Prop. V.).

Cor. If the projection be that of a great circle, becauſe CL is perpendicular to AU, it will de perpendicular alſo to KT which is parallel to AU (1. Cor. Prop. XI.); wherefore the projected pole will be found ſomewhat eaſier thus. Draw AT as before, and joining TC, draw CL at right angles thereto, and LP perpendicular to the line of meaſures, P ſhall be the projected pole.

PROP. XIV. PROBLEM.

A PORTION of a projected circle being given, to find a ſimilar arch of the primitive; and Converſely, from a given projected circle and at a given point to cut off a portion ſimilar to a given arch of the primitive.

Case 1. When the given projection is that of a great circle.

Let RDS be a projection rectilinear, or elliptical, of a great circle, and DE a given portion of the projection, it is required to find an arch of the primitive equal to the arch of which DE is the repreſentative. Draw the line of meaſures CG (Prop. XII.), and DF, ET parallel thereto, meeting the primitive in F, T. Then FT is an arch of the primitive equal to the arch of the great circle repreſented by DE (2. Cor. Prop. VII.). Fig. 45. 47.

And Converſely, If it be required at a point D to cut off a portion of the projected great circle RDS, which ſhall repreſent an arch equal to a given arch GL of the primitive; draw DF parallel to the line of meaſures CG, and meeting the primitive in F. In the primitive make the arch FT equal to GL, and draw TE parallel to CG, DE ſhall be the portion required. For DE repreſents an arch equal to FT (2. Cor. Prop. VII.) of the primitive, and therefore to GL.

Case 2. When the projection is of a leſs circle, perpendicular, or inclined to the primitive.

Let RDS be a rectilinear, or elliptical projection of a leſs circle, 48. 51.

circle, and DE a portion thereof; it is required to find an arch of the primitive ſimilar to the arch of which DE is the repreſentative. Draw COG the line of meaſures, which will paſs through O the centre of the projection, biſecting RS the whole rectilinear projection or RS the major axis of the ellipſe. Wherefore round O with the diſtance OR or OS deſcribe a circle, and let DF, ET parallel to CO, and meeting this circle in F, T, be drawn. FT will be an arch of the circle RFS ſimilar to the arch repreſented by DE (1. COR. PROP. VII.), and therefore equal alſo, becauſe the diameter of the circle RFS is a diameter alſo of the circle of the ſphere repreſented by the projection RDS (PROP. II. and V.). Wherefore draw OF, OT, and from the centre C of the primitive CG, CL containing an angle GCL equal to the angle FOT. GL the arch of the primitive intercepted by them is ſimilar to the arch FT (LEM. VIII.), viz. to the arch repreſented by DE.

AND Converſely, If an arch GL of the primitive be given, and it be required at the given point D in the given projection RDS to cut off a portion which ſhall repreſent an arch ſimilar to GL. The line of meaſures, and the circle RFS, as alſo CG, CL being drawn as before, draw DF parallel to CO meeting the circumference RFS in F, join OF, and draw OT making with OF the angle FOT equal to the angle GCL, alſo draw TE parallel to OC meeting RDS in E.

DE ſhall be the portion required. For in the circle RFS and the primitive, the arches FT, GL are ſimilar, becauſe ſubtending equal angles. But the arch FT is equal to the arch of which DE is the repreſentative, therefore DE is a portion of the given projection repreſenting an arch ſimilar to the given arch GL of the primitive.

Case 3. When the given projection is that of a circle parallel to the primitive.

Let RDS be a given projection parallel to the primitive, and DE a given portion thereof. From C the centre of the primitive draw CD, CE, meeting the primitive in F, T. Then FT is an arch of the primitive ſimilar to DE. *Fig.* 50.

Or if GL be a given arch of the primitive, and from a given point D in the parallel projection it be required to cut off a portion ſimilar to GL. Draw CD, meeting the primitive in F, make the arch FT equal to GL, join CT meeting the projection in E, and the arch DE will be a portion ſimilar to FT or GL.

PROP. XV. PROBLEM.

THROUGH two given points in the plane of the primitive to deſcribe the projection of a great circle.

CASE 1. IF the two given points and the centre of the primitive be in the ſame right line, then this right line anſwers the conditions of the Problem, being the projection of a right circle.

CASE 2. WHEN the two given points are not in the ſame right line with the centre of the primitive, but one of them is in the circumference of the primitive.

Fig. 46. LET D, R be the two given points, of which R is in the circumference of the primitive. Draw RCS through the centre C of the primitive, and GCg, FDH at right angles to it, meeting the primitive in G, g, F. Divide GC, gC in A, B in the ſame proportion as FH is divided in D, and deſcribe the ellipſe whoſe axes are RS, AB, and centre C. This ellipſe will repreſent a great circle, becauſe its axis major RS is a diameter of the primitive, and it will paſs through the given point D. For if not, yet becauſe FH parallel to AB falls within the ellipſe, it muſt meet the ellipſe on each ſide of RS, viz. in one point towards the ſame parts with D; if poſſible, therefore, let it meet it in O, then as GC is to AC, ſo is FH to OH (1. Cor. Prop. IV.). But, by hypotheſis, as GC is to AC, ſo is FH to DH; wherefore OH is equal to DH, a part to the whole, which is abſurd. Therefore FH does not meet the ellipſe towards the parts of D in any point unleſs D only, viz. the ellipſe paſſes through D.

Case 3. When both the given points are within the primitive, and not in the ſame right line with the centre.

Let D, E be the two given points, and IDi, KEk be drawn through C the centre of the primitive; draw DL at right angles to Ii, and EO to Kk, meeting the primitive in L, O. Through E and towards the parts of C draw EP parallel to DC, and in magnitude a fourth proportional to LD, DC, OE. Draw the diameter CP meeting the primitive in R, S, and by the preceding caſe deſcribe an ellipſe paſſing through the points D and R or S; I ſay, it will paſs through E alſo. Draw the axis minor AB, meeting the primitive in G, g, and DH, EQ parallel to AB meeting the primitive in F, f, T, t, and the axis major in H, Q. *Fig.* 45.

BECAUSE LD is to DC as OE is to EP, and from ſimilar triangles, DC is to DH as EP is to EQ, therefore ex æquo, LD is to DH as OE is to EQ, and the ſquare of LD is to the ſquare of DH as the ſquare of OE to the ſquare of EQ. But from the property of the circle (PROP. IV. B. I.), the ſquare of LD is equal (to the rectangle IDi, viz.) to the rectangle FDf, and for the ſame reaſon, the ſquare of OE is equal to the rectangle TEt. Therefore the rectangle FDf is to the ſquare of DH as the rectangle TEt is to the ſquare of EQ, and by compoſition, the rectangle FDf together with the ſquare of DH is to the ſquare of DH as the rectangle TEt together with the ſquare of EQ is to the ſquare of EQ. But becauſe Ff, Tt are perpendicular to RS, they are biſected in H, Q, wherefore the rectangle FDf together with the ſquare of DH is equal to the ſquare of FH, and the rectangle TEt together with the ſquare of EQ is equal to the ſquare of TQ (5. E. 2.). Therefore the ſquare of FH is to the ſquare of DH as the ſquare of TQ is to the ſquare of EQ, and FH is to DH as TQ to EQ. But becauſe D is a point in the ellipſe, FH is to DH as GC to AC; wherefore as GC to AC ſo is TQ to EQ, and ſo in the ſame manner as in the preceding caſe is the point E ſhewn to be in the ellipſe.

PROP. XVI. PROBLEM.

THROUGH a given point in the plane of the primitive, but not without the circumference, to deſcribe the projection of a great circle, which ſhall have a given inclination to the primitive. But it is neceſſary that the given point be not at a diſtance from the centre of the primitive leſs than the coſine of the given inclination.

Case 1. When the given inclination is that of a right angle. As the projection will be of a right circle, a right line drawn through the centre of the primitive and the given point will be the projection required.

Case 2. When the given inclination is leſs than a right angle, but the given point is in the circumference of the primitive.

Let R be a given point in the circumference of the primitive, through which it is required to draw the projection of a great circle inclined to the primitive in an angle meaſured by the given arch QP of the primitive. *Fig.* 46.

DRAW the diameter RCS of the primitive, and GCg at right angles to it. Make the arch GV of the primitive equal to QP, and draw VA at right angles to GC, and in Gg towards the oppoſite parts of C taking CB equal to AC, deſcribe the ellipſe whoſe axis major is RS, and minor AB; this ellipſe ſhall be the projection required.

FOR becauſe AC the ſemiaxis minor is the coſine of an arch GV equal to the given arch QP meaſuring the given inclination, therefore the ellipſe RAS is the projection of a great circle having the given inclination to the primitive (5. COR. PROP. V.); and it is drawn through the given point R.

CASE 3. WHEN the given point is at a diſtance from the primitive equal to the coſine of the given inclination.

THUS every thing remaining as in the preceding caſe, if A be the given point, and AC be the coſine of an arch GV equal to the given arch QP, then drawing the diameter RCS at right angles to ACB, the ellipſe deſcribed round the centre C with the given axes RS, AB will anſwer the conditions of the Problem.

CASE 4. WHEN the given point is at a diſtance from the centre of the primitive leſs than the ſemidiameter of the primitive, but greater than the coſine of the given inclination.

Let D be the given point, through which draw the diameter ICi, and at the point D draw DL perpendicular to DC meeting the primitive in L, alſo LK making with LD the angle DLK equal to the complement of the given inclination. Let LK meet DC in K, then will DK be leſs than DC. For QP being the arch of the primitive meaſuring the given inclination, join CQ, CP, and draw PT perpendicular to CQ. Then CT is the coſine of the given inclination, and conſequently leſs than CD, wherefore joining CL, becauſe CL is equal to CP, the ſquares of CD, DL are together equal to the ſquares of CT, TP, but the ſquare of CD is greater than the ſquare of CT, therefore the ſquare of DL is leſs than the ſquare of TP, and DL itſelf is leſs than TP. But becauſe the angle CPT is the complement of the angle PCQ, viz. of the given inclination, the right-angled triangles CPT, KLD are equiangular, wherefore PT is to TC as LD to DK, and becauſe LD has been ſhewn to be leſs than PT, therefore DK is leſs than TC (14. E. 5.). But TC itſelf is leſs than DC, and conſequently DK is much leſs than DC. On DC as a diameter deſcribe a circle, and therein inſcribe DH equal to DK (1. E. 4.), draw through H a diameter of the primitive RCS, and (as in Case 2.) deſcribe an ellipſe through the point R or S which ſhall repreſent a great circle having the given inclination to the primitive; this ellipſe ſhall paſs through D. Let AB be the minor axis of this ellipſe at right angles to the major RS, and let it meet the primitive in G, g; alſo produce QC meeting the primitive in q, and DH to meet it in F, f. Becauſe the angle DHC in a ſemicircle is a right angle, Ff is parallel to AB, and therefore biſected in the point H; and becauſe the elliptical projection repreſents a circle inclined to the primitive in an angle equal to the angle QCP, therefore the ſemiaxis minor AC is equal to CT the coſine of this angle (5. COR. PROP. V.), and conſequently the rectangle GAg is equal (to the rectangle QTq, viz.) to the ſquare of PT (PROP. IV. B. I.). Alſo the rectangle FDf is equal (to the rectangle IDi, viz.) to the ſquare of DL, and DH is equal to DK. But it has been ſhewn that PT is to TC as LD to DK, or that the ſquare of PT is to the ſquare of TC as the ſquare of LD is to the ſquare of DK, and conſequently the magnitudes now ſhewn to be equal to them will alſo be proportional, viz. the rectangle GAg is to the ſquare of AC as the rectangle FDf is to the ſquare of DH. Therefore, by the ſame reaſoning as in CASE 3. PROP. XV., the point D is ſhewn to be in the ellipſe.

SCHOL. THIS is only one caſe of the general Problem, viz. through a given point to deſcribe the projection of a great circle, which ſhall have a given inclination to a given great circle. But as the Orthographic Projection is very properly rejected in the conſtruction of thoſe Problems where elliptical projections would generally be required, it is unneceſſary to give the ſolution of all the other caſes of this Problem.

But to gratify the curioſity of the reader, one manner of a general ſolution of it will be taken notice of when the ſame Problem in the Stereographic Projection comes under conſideration.

PROP. XVII. PROBLEM.

About a given point, not without the primitive, as a projected pole, to deſcribe the projection of a circle, whoſe diſtance from its pole ſhall be equal to a given arch of the primitive.

Case I. When the given point is in the centre of the primitive.

The projection in this caſe is a circle, whoſe centre is the centre of the primitive, and ſemidiameter the ſine of its diſtance from the pole of the primitive, viz. from its own pole (PROP. III. and COR. of the ſame.). But this diſtance is given, therefore the circle itſelf is given.

CASE 2. WHEN the point is given in the circumference of the primitive.

THE projection in this caſe will be a right line, as the circle to be repreſented by it muſt be perpendicular to the primitive (PROP. III. B. II.).

CASE 3. WHEN the given point is within the primitive, but not in the centre.

Fig. 43. 44. LET P be the given point not without the primitive, through which and C the centre of the primitive draw the diameter GCg, which will be the line of meaſures of the circle to be projected (5. COR. PROP. V.). Draw PL, CH perpendicular to GC meeting the primitive in L, H; make the arches LT, LK equal to the given arch of the primitive, and join CL, TK meeting each other in Q, draw QOS perpendicular to GC, and therein, on each ſide of the point O its concourſe with GC, make OR, OS equal to QT, or QK, alſo draw TA, KB perpendicular to GC. The ellipſe deſcribed round the centre O whoſe axes are AOB, ROS, of which ROS is the major, will be the projection required.

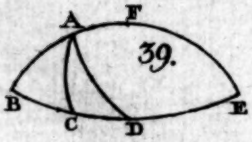
For the ſemidiameter of the circle repreſented by the elliptical projection ARB is equal (to OR the ſemiaxis major (Prop. V.), viz.) to TQ. But the ſemidiameter of every circle of the ſphere is the ſine of its diſtance from its own pole, and TQ is the ſine of the arch TL, becauſe it is perpendicular to the diameter CL; therefore the diſtance of the circle from its pole is equal to the arch TL. And TL is equal to the given diſtance, therefore the projection ARB repreſents a circle of the ſphere, whoſe diſtance from its pole is equal to the given arch. Alſo in the ſame manner as in Prop. XIII. is P ſhewn to be the projected pole of the circle repreſented by ARB.

OF THE
STEREOGRAPHIC PROJECTION
OF THE
SPHERE.

BOOK V.

PROP. I. THEOREM.

A CIRCLE of the ſphere paſſing through the projecting point, is projected into a right line, which is the common ſection of the plane of the circle with the primitive, and is perpendicular to the line of meaſures of the circle.



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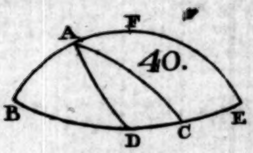

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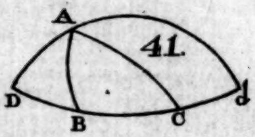

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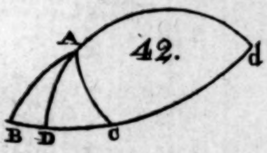

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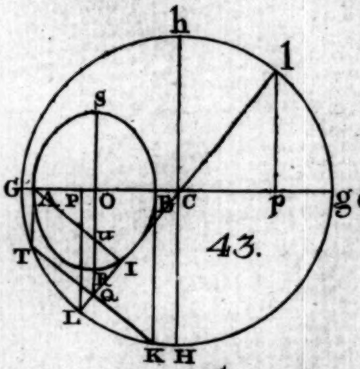

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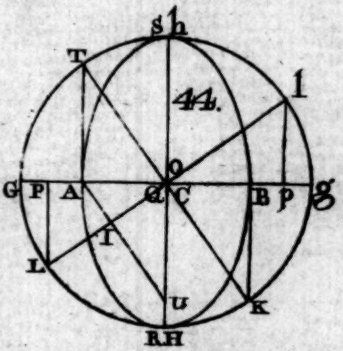

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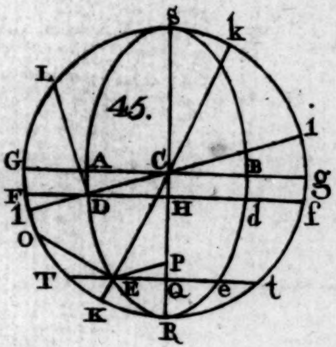

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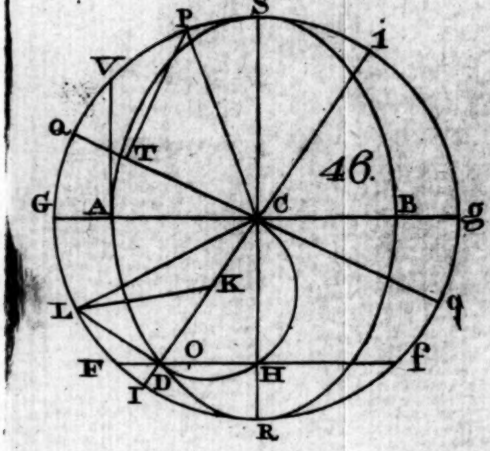

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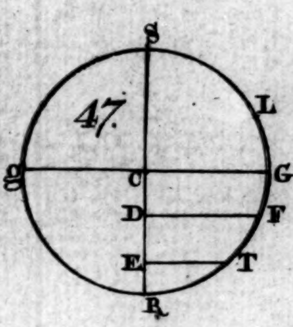

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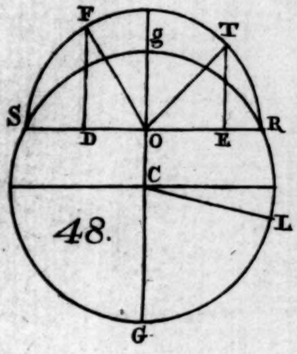

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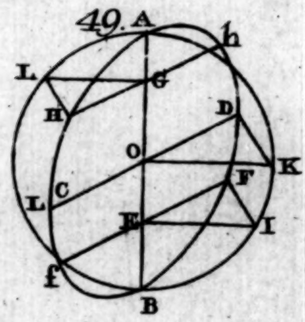

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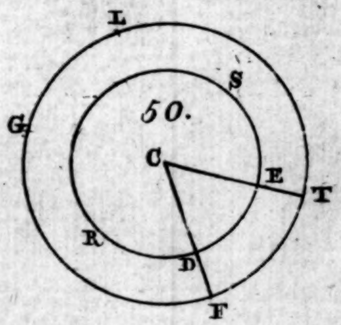

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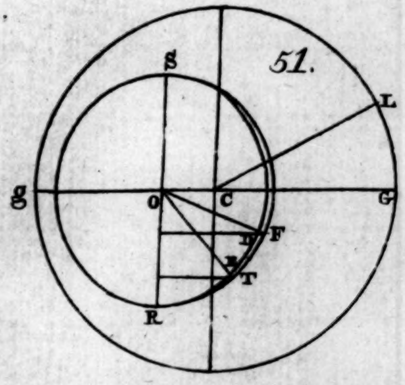

51.

TAB. V. *pag.* 165.

For all the right lines iſſuing from the projecting point to the circle's periphery are in the plane of the circle (7. Def. E. 1.), and therefore can meet the plane of projection only in the common ſection of the two planes. But this common ſection is a right line (3. E. 11.). Therefore the circle is projected into a right line, which is the common ſection of the circle with the plane of projection, and therefore perpendicular to its line of meaſures (3. Cor. Def. B. IV.).

Cor. A great circle paſſing through the projecting point, is projected into a right line paſſing through the centre of the primitive.

For the primitive being alſo a great circle, their common ſection is a diameter of the ſphere (Prop. II. B. II.), and therefore of the primitive alſo.

PROP. II. THEOREM.

EVERY circle of the ſphere, not paſſing through the eye, or the projecting point, is projected into a circle.

CASE I. WHEN the circle to be projected is parallel to the primitive.

BECAUSE the rays of viſion iſſuing from the eye to the circumference of the circle form a cone, the projection of the circle will be the line bounding the ſection which the plane of the primitive makes with this cone. But the plane of the primitive is parallel to the circle which is the baſe of the cone (HYP.), therefore the ſection of the cone, viz. the projection of the circle, is itſelf a circle (PROP. III. B. I.).

Fig. 22. THUS the circle BTP parallel to the primitive EaD is projected into a circle btp.

20. 21. CASE 2. WHEN the circle to be projected, as BTP, is oblique, viz. inclined to the primitive EaDi.

LET A be the projecting point, viz. the pole of the primitive in which the eye is ſituated. Then, as in the preceding caſe, the rays of viſion form a cone whoſe baſe is the circle BTP, and the ſection of this cone by the primitive is the projection of BTP. Let btp be this ſection; I ſay, btp is alſo a circle.

To C, O the centres of the ſphere and of the circle BTP draw AC, CO. Becauſe the primitive EaD and the circle

BTP are not parallel to each other, the ſame right line is not perpendicular to each of their planes (14. E. 11.), wherefore CO which is perpendicular to BTP (8. DEF. and 3. COR. PROP. I. B. II.) is not perpendicular to EaD, and conſequently does not paſs through A the pole of Ead (8. DEF. B. II.); viz. AC, CO are not in one ſtraight line. ACO is therefore a plane, which will cut the ſphere in a great circle EIDA, the primitive and the circle BTP in the diameters ED, BP (PROP. II. and III. B. II.), the cone BTPA in the triangle BAP (PROP. II. B. I.), and let it cut the projection in the right line bp. Join AO, becauſe CO is perpendicular to the plane BTP, AO is not perpendicular thereto (13. E. 11.); but AO is the axis of the cone, therefore the cone is a ſcalene cone (7. DEF. B. I.). And becauſe CO is at right angles to BTP, the triangular ſection BAP, which paſſes through CO, is perpendicular to BTP the baſe of the cone (18. E. 11.). Draw BN parallel to ED, becauſe AC is perpendicular to the plane EaD, it will be perpendicular to the right line ED (3. DEF. E. 11.), and conſequently to BN, wherefore the arch BAN is biſected in A (30. E. 3.), and the angle BPA is equal (to the angle ABN (27. E. 3.), viz. becauſe of the parallels BN, ED) to the angle Abp. Therefore the triangles ABP, Abp, having alſo a common angle at A, are equiangular, the angles at their baſes BP, bp, taken alternately, being equal between themſelves; conſequently,

quently, the cone BTPA is cut ſubcontrarily by the primitive (9. DEF. B. I.), and therefore the ſection, i. e. the projection btp is a circle (PROP. V. B. I.).

BY the very ſame reaſoning, if the circle were a great one, as BaHi, would it be ſhewn that bahi the projection thereof is a circle. For BN being drawn parallel to ED, it is in the ſame words ſhewn that the angle AHB is equal (to the angle ABN, viz.) to the angle Abh, and therefore, &c.

COR. 1. THE line of meaſures of any circle of the ſphere paſſes through the centre of the projection.

FOR the triangular ſection of the cone being a right circle, and perpendicular to the baſe of the cone, which is the circle to be projected, cuts the primitive in the line of meaſures (8. DEF. B. IV.), and alſo in the ſame right line cuts the ſubcontrary ſection. But this right line is a diameter to the ſubcontrary ſection (PROP. V. B. I.), therefore, &c.

COR. 2. THE projection of a circle parallel to the primitive has the ſame centre with the primitive itſelf, and its radius is the ſemitangent of the circle's diſtance from the oppoſite pole of the primitive.

For the circle and the primitive, being parallel, have the ſame poles (3. Cor. Def. B. II.), wherefore the right line AI joining the common poles A, I is perpendicular to each, and paſſes through their centres C, O (3. Cor. Prop. III. B. II.). The pole A being therefore the projecting point, the centre O is projected in C, and ACO being perpendicular to EC, the radius Cb of the projection is the ſemitangent of the arch BI, which is the diſtance of the circle BTP from the oppoſite pole I. *Fig.* 22.

Cor. 3. If a right circle perpendicular to any circle BTP of the ſphere cut the ſame in the diameter BP, and the primitive in ED, and the right lines AbB, ApP be drawn, bp which is the projection of the diameter BP will make angles with AB, AP equal to the alternate angles which BP makes with the ſame.

Cor. 4. The cone of rays btpA is likewiſe a ſcalene one, and BTP a ſubcontrary ſection of it. 20. 21.

For becauſe the arches AP, AN, and therefore AP, AB are unequal, the angles ABP, APB, or the angles Apb, Abp will be unequal. Therefore Ao the axis of the cone is not at right angles to the baſe btp. For if it be perpendicular thereto, it will be perpendicular to the diameter bp, wherefore bo being equal to op, Ao common, and the angle Aob equal to the angle Aop, the homologous angles Abp, Apb would be equal between themſelves, which is abſurd. Therefore the cone is ſcalene, and the triangle ABP is equiangular to the triangle Abp in a ſubcontrary poſition.

PROP. III. THEOREM.

THE interſections of every projected great circle with the primitive are in a diameter to the primitive, at right angles to the line of meaſures of the projection.

FOR the primitive cuts every great circle in a right line which is a diameter of each, as alſo of the ſphere itſelf (PROP. II. B. II.). Therefore the extremities of this diameter being the ſame on the ſphere, as on the plane of projection, the projection cuts the circumference of the primitive in two points which are the extremities of a diameter to the primitive. But this diameter being the common ſection of the great circle with the primitive, will be perpendicular to the line of meaſures (3. COR. DEF. B. IV.).

Fig. 20. THUS the ſame things remaining as in the preceding Propoſition, the projection bahi of the great circle BaHi paſſes through the points a, i which are the extremities of aCi the common diameter of the primitive and the circle BaHi. And becauſe the great circle BaHi and the primitive are both perpendicular to the right circle EID (8. Def. B. IV.), their common ſection aCi will be perpendicular to the plane EID (19. E. 11.), and therefore will be perpendicular to the line of meaſures ED (3. Def. E. 11.).

Cor. By the ſame reaſoning it appears univerſally, that if any circle of the ſphere cut the primitive, its projection will cut the primitive in the ſame right line which will be perpendicular to the line of meaſures.

PROP. IV. THEOREM.

EVERY point on the ſurface of the ſphere is projected at a diſtance from the centre of the primitive equal to the ſemitangent of the point's diſtance from the pole of the primitive oppoſite to the projecting point.

The ſame things remaining as in Prop. II., if P be any point on the ſurface of the ſphere, and p its projection; IP is the diſtance on the ſurface of the ſphere of the point P from I the oppoſite pole of the primitive, and Cp which is Fig. 20.

is the diſtance of the point p from the centre C of the primitive is the ſemitangent of the arch IP.

COR. 1. IF p be the projection of any point on the ſurface of the ſphere, and the diameter Cp of the primitive be drawn, and aCi at right angles to Cp, alſo ap*p* meeting the circumference of the primitive in *p*; the arch i*p* ſhall be equal to the diſtance on the ſurface of the ſphere of the point from the pole oppoſite to the projecting point.

FOR the ſame right line Cp is the ſemitangent of the arches IP, i*p*, and the circles, of which they are arches, are equal; therefore the arches themſelves are equal.

COR. 2. THE line of meaſures of every circle of the ſphere, unleſs a right one, cuts its projection in two points which are diſtant from the centre of the primitive by the ſemitangents of the circle's greateſt and leaſt diſtances from the oppoſite pole of the primitive.

FOR every thing remaining as in PROP. II., IB, IP are the greateſt and leaſt diſtances of the point I from the circle BTP (COR. PROP. V. B. II.), and Cb, Cp are the ſemitangents of the arches IB, IP.

Fig. 20. 21. COR. 3. THE diameter of the projection is the ſum of theſe ſemitangents, if the circle encompaſs the oppoſite pole of the primitive, but if the pole lie without the circle, it is equal to the exceſs of the greater above the leſs.

Cor. 4. The centre of the projection is at a diſtance from the centre of the primitive equal to half the difference, or half the ſum of theſe ſemitangents, accordingly as the oppoſite pole of the primitive is ſituated within or without the circle.

Cor. 5. If the circle be a great one, the ſemitangents of the diſtances from the pole of the primitive are the ſame with the tangent and cotangent of half the circle's inclination to the primitive.

For the leaſt diſtance of the great circle from the pole of the primitive, is the complement of the circle's inclination to the primitive.

PROP. V. THEOREM.

If a diameter of the primitive be drawn at right angles to the line of meaſures of any circle of the ſphere, and from either extremity of this diameter right lines be drawn to the interſections of the line of meaſures with the circle's projection; the arch of the primitive intercepted by the right lines ſo drawn ſhall be double to the circle's diſtance from its nearer pole, the right line ſubtending this arch ſhall be equal to the diameter of the circle, and have the ſame inclination to the line of meaſures that the circle has to the plane of the primitive.

Fig. 20. 21. EVERY thing remaining as in PROP. II., let a diameter aCi of the primitive be drawn at right angles to bp the the line of meaſures of the circle BTP, or of its projection btp, and let ab, ap be drawn meeting the circumference of the primitive in *b*, *p*; I ſay, the arch *bp* ſhall be double to the diſtance of the circle BTP from its nearer pole R, viz. ſhall be equal to the arch BIP, and *bp* being joined ſhall be equal to BP the diameter of the circle BTP, and ſhall have the ſame inclination to ED the line of meaſures which the circle BTP has to the primitive.

BECAUSE aC is equal to AC, bC is common, and the angle aCb equal to the angle ACb being each right, the correſpondent angles baC, bAC are equal (4. E. 1.). For the ſame reaſon, is the angle paC equal to the angle pAC, and conſequently the whole bap equal to the whole angle bAC. But in equal circles equal angles ſtand upon equal circumferences (26. E. 3.), and equal circumferences are ſubtended by equal right lines (29. E. 3.), wherefore the circumference *bip* is equal to the circumference BIP, and and *bp* to BP. Again, becauſe it has been ſhewn that the angle *b*ai is equal to the angle BAI, the arch *b*i is equal to the arch BI, as for the ſame reaſon is the arch *p*i equal to the arch PI. Draw *b*n parallel to ED, becauſe ai is drawn through the centre perpendicular to ED, viz. to *b*n, the arch *b*i is equal to the arch ni (30. E. 3.), and for the ſame reaſon is the arch BI equal to the arch NI. But the arches *b*i, BI are equal, therefore the arches ni, NI are equal, and the parts *p*i, PI have been ſhewn to be equal, therefore the remaining arches n*p*, NP are equal between themſelves, and being arches of equal circles, the angles n*bp*, NBP are alſo equal (27. E. 3.). But the angle n*bp* is the inclination of the right line *bp* to the right line *b*n, viz. to its parallel ED, and becauſe the arch NP meaſures the inclination of the plane BTP to the plane EaD of the primitive (Prop. VI. B. II.), the angle NBP is equal to this inclination. Wherefore the inclination of *bp* to the line of meaſures ED is equal to the inclination of the circle BTP to the primitive.

In the ſame words is the Propoſition demonſtrated, if the circle were a great one, as BaHi.

Cor. 1. If from either interſection of a projected great circle with the primitive right lines be drawn to the interſections of the projection with the line of meaſures, theſe will meet the circumference of the primitive in two points which are in a diameter to the primitive, and the angle which this diameter makes with the line of meaſures will be equal to the inclination of the great circle to the primitive.

Cor. 2. If from either interſection of a diameter to the primitive at right angles to the line of meaſures of any circle, right lines be drawn through the interſections of the projection with the line of meaſures, the arches of the primitive intercepted between each of theſe right lines and the diameter of the primitive ſhall be equal to the circle's leaſt and greateſt diſtances from the oppoſite pole of the primitive.

For the arches *ip*, *ib* have been ſhewn to be equal to IP, IB, which are the leaſt and greateſt diſtances of the point I from the circle BTP (Cor. Prop. V. B. II.).

Cor. 3. If the projection be that of a leſs circle perpendicular to the primitive, the right lines drawn from either extremity of the diameter of the primitive at right angles to the line of meaſures, through the interſections of the line of meaſures with the primitive, will paſs through the interſections of the projection with the circumference of the primitive.

LET a leſs circle SRTV perpendicular to the primitive be projected into the circle SrTv, cutting the primitive in the points S, T, and the line of meaſures ED in r, v; the points a, r, T, and a, v, S will be in a ſtraight line. *Fig.* 29.

FOR the points T, S are the ſame with the points on the ſphere repreſented by them, and E, D the points in which the line of meaſures meets the primitive, are the poles of the circle SRTV (7. and 5. COR. DEF. B. IV.). Therefore DT, DS are the diſtances of the circle from its nearer pole D. But by this Propoſition, the right lines ar, av will intercept an arch of the primitive double to the diſtance of the circle from its nearer pole, viz. equal to TS, and the right line ſubtending the intercepted arch will be perpendicular to ED. Wherefore the intercepted arch will be biſected in D (30. E. 3.), that is, the right lines ar, av will meet the primitive on each ſide of D at diſtances equal to DT, DS, and therefore in the points T, S themſelves.

COR. 4. HENCE therefore if the line of meaſures of a circular projection be given, and the right line bp in poſition and magnitude, every thing relating to the circle repreſented by the projection, is projected from the primitive and 20. 21.

the

the point a, as from the right circle AEID, and the pole A of the primitive.

FOR the points a, b, *b*, and a, p, *p*, being in a right line, the points b, p, and the diameter bp of the projection are given. Therefore the centre o, viz. the point in which bp is biſected, and conſequently the projection itſelf is given. And if *bn* parallel to ED be drawn, becauſe *bp* has the ſame inclination to ED that it has to *bn*, and the inclination of *bp* to ED is equal to the inclination of the circle to the primitive, therefore the angle *pbn*, meaſuring the inclination of the circle to the primitive, is given.

ALSO, r being the projection of the pole R, if a*r* be joined, it will biſect the arch *bp* in the concourſe F, and therefore the point F being given the projected pole *r* is given. For in the ſame manner as in the propoſition, is it ſhewn that the arches F*b*, F*p* are equal to the arches RB, RP. But RB, RP are equal between themſelves, therefore alſo F*b*, F*p* are equal, viz. the whole arch *bp* being given in poſition and magnitude, the point F in which it is biſected, is given, and therefore the projected pole *r*.

AND for the ſame reaſon, if from a to the projected place of the other pole r, a right line were drawn, the arch bap would be biſected thereby in the concourſe f, and therefore this arch, and the point of biſection being given, the other projected pole is given.

Cor. 5. The arches *i*F, i f are the meaſures of the circle's inclination to the primitive, and the whole intercepted circumference FIf is a ſemicircle. For they are equal to the arches IR, Ir, which are the meaſures of the circle's inclination. (P. VI. B. II.), and together equal to half the great circle EID, becauſe Rr is a diameter thereof.

PROP. VI. THEOREM.

A RIGHT line touching a circle of the ſphere and intercepted by the primitive is projected into an equal right line, and if the circle be not a right one, it will touch the circular projection in the point repreſenting the point of contact on the ſphere.

Case 1. When the circle is a great one, and paſſes through the projecting point.

Let AEP be a great circle paſſing through the projecting point A, and PV a right line touching it in the point P, and meeting the primitive in V; draw AP cutting the primitive in p; I ſay pV is equal to PV. *Fig.* 24.

Because

BECAUSE AEP is a great circle, and pV is its common ſection with the primitive, pv will paſs through the centre C of the ſphere. Let it meet the primitive in E, D and draw PN parallel to ED, alſo join AC, AN. Becauſe AC is perpendicular to the primitive (8. DEF. B. II.), it will be perpendicular to ED and therefore to the parallel PN, and being drawn through the centre of the circle AEP, it will biſect PN, and conſequently the arch PAN (3 & 30. E. 3.). Wherefore the angle APN is equal to the angle ANP (27. E. 3.), but becauſe of the parallels pV, PN, the angle PpV is equal to the angle APN, and becauſe PV touches the circle in P, the angle pPV is equal to the angle ANP in the alternate ſegment (32. E. 3). Wherefore the angle PpV is equal to the angle pPV, and conſequently the ſide pV is equal to the ſide PV (6. E. 1.).

CASE 2. LET FPG be any other circle, FpG its projection, PT a right line touching the circle in the point P, and meeting the primitive in T; I ſay pT, which is the projection of PT is equal to PT, and touches the projection FpG in p.

C being the centre of the ſphere, and A the projecting point, join AC, CP. Becauſe p is the projected place of the point P; A, p, P are in one ſtraight line (1. DEF. B. IV.), which let be drawn. Let the plane ACP paſſing through the centre of the ſphere cut it in the great circle AEPD, and the primitive in the right line ED.

DRAW PV touching the circle EPD in P, and let it meet the primitive in V, alſo join TV. The plane TPV will touch the ſphere in P, for if not, then the plane which does touch it in P, would cut the circles FPG, EPD in other right lines than PT, PV, and theſe right lines would alſo touch the circles in P (PROP. X. B. II.), which is abſurd (COR. 16. E. 3.). Wherefore no other plane than TPV touches the ſphere in P, and conſequently it is perpendicular to the plane EPD (COR. PROP. IX. B. II.). But the primitive is alſo perpendicular to the plane EPD (PROP. III. B. II.), therefore TV, which is the common ſection of the primitive with the plane TPV is perpendicular to the plane EPD (19. E. 11.), and conſequently to the right lines VP, Vp meeting it in that plane. But Vp is equal to VP, by the preceding caſe; therefore in the triangles VpT, VPT, becauſe Vp is equal to VP, and VT common, and the angle pVT equal to the angle PVT, the baſe pT is equal to the baſe PT (4. E. 1.). Therefore PT is projected into an equal right line pT. And becauſe PT touches the baſe FPG of the cone AFPG, the plane APT will touch the cone in the right line AP (PROP. VI. B. I.). Wherefore the right line pT which is drawn in the touching plane can meet the ſuperficies of the cone only in the point p, and conſequently in that point only can meet the circumference of the circular projection FpG which is ſituated in that ſuperficies, viz. pT touches the circle FpG in p.

Cor. 1. If two circles of the ſphere cut each other, and two right lines touch them at their interſection, the plane paſſing through the touching lines will touch the ſphere.

Cor. 2. If any number of circles of the ſphere cut each other in the ſame point, the right lines touching each in their common concourſe, will all be in one and the ſame plane, viz. in the plane touching the ſphere in the ſame point of concourſe.

For every two of them are in the plane touching the ſphere in the point of concourſe, but this plane is one and the ſame to all, becauſe only one plane can touch the ſphere in the ſame point (Cor. Prop. VII. B. II.).

PROP. VII. THEOREM.

An angle comprehended by two great circles on the ſurface of the ſphere is equal to the angle comprehended by their repreſentatives on the plane of projection.

CASE 1. WHEN the great circles are both right.

LET EID be the plane of projection, a, i its poles, and aEi, aBi two right circles, projected into the diameters ECD, BCH of the primitive; I ſay the ſpheric angle EaB is equal to the angle ECB. *Fig.* 20.

BECAUSE a is the pole of the great circle EID, the arch EB meaſures the ſpheric angle EaB (6. COR. PROP. I. B. III.), and the ſame is the meaſure of the angle ECB (3. DEF. B. III.) therefore the angle EAB is equal to the angle ECB.

CASE 2. WHEN one of the great circles is right, but the other inclined obliquely to the primitive.

THE ſame things remaining as in PROP. VI., I ſay the ſpheric angle DPG is equal to its repreſentative DpG. 24.

FOR becauſe VP is equal to Vp, TP to Tp, (PROP. VI.) and VT common to each of the triangles VPT, VpT, the angle VPT is equal to the angle VpT (8. E. 1.). But the angle VPT is the meaſure of the ſpheric angle DPG (Ax. B. III.), and the angle VpT is the meaſure of the angle DpG*, therefore the ſpheric angle DPG is equal to the angle DpG.

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* IF a right line cut a circle, or two circles cut each other, the angle comprehended by the right line and the circumference is meaſured by the right-lined angle

CASE 3. WHEN both the great circles are inclined obliquely to the primitive.

THE ſame things remaining; I ſay, the ſpheric angle GPK is equal to the angle GpK comprehended by the circles repreſenting GP, PK.

FOR by the ſame reaſoning as in the preceding caſe, is it ſhewn that the angles TPW, TpW, and therefore that the angles GPK, GpK, are equal between themſelves.

Fig. 53. COR. THE ſame things remaining, if O, Q be the centres of the projections of two great circles, and to either of their interſections P be drawn OP, QP, the angle OPQ ſhall be equal to the ſpheric angle comprehended by the great circles of the ſphere repreſented by GP, PK.

LET

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angle comprehended by the right line and the tangent to the circle at the point of concourſe: and, the angle comprehended by the circumferences is meaſured by the right-lined angle comprehended by the tangents at the concourſe.

Fig. 53. LET the right OP cut the circle KP in the point P, and PW be drawn touching the circle, the angle OPK is meaſured by the right-lined angle OPW. The angle KPW is leſs than any right-lined angle which can be formed (16. E. 3.), and therefore the angle OPW which exceeds the angle OPK only by the angle KPW, is juſtly aſſumed as the meaſure of the angle OPK.

AND, if two circles KP, GP cut each other in P, and PW, PT be drawn touching the circles, the angle WPT ſhall be the meaſure of the angle KPG. Draw any right line OP. The angle OPW meaſures the angle OPK, and the angle OPT meaſures the angle OPG, therefore the remaining angle WPT meaſures the remaining angle KPG.

LET Tp, Wp touch the circles in p, then the angles OpT, QpW are equal being each a right angle (18. E. 3.), therefore take away the common angle OpW, and the remaining angle OpQ ſhall be equal to the remaining angle TpW, viz. to the angle GpK, and therefore to the ſpheric angle repreſented by GpK. *Fig.* 53.

COR. 2. AND by the ſame reaſoning, if two circles touch each other on the ſurface of the ſphere, will their projections touch each other on the plane of the primitive.

FOR becauſe the common ſection of the circles touches each of them on the ſurface of the ſphere in the point of mutual contact (PROP. XI. B. II.), the projection of this common ſection will be a tangent to each of the projections in the point repreſenting their point of contact on the ſphere (PROP. VI.). Wherefore the ſame right line touching each of the projections in the ſame point, they will touch each other in that point.

COR. 3. IF two great circles on the ſurface of the ſphere be perpendicular to each other, their projections will alſo be perpendicular to each other.

COR. 4. IF the angular point of a projected ſpheric angle be the pole of a projected great circle, the arch intercepted by the ſides of the projected angle will have the ſame meaſure with the angle itſelf.

For the arch on the ſurface of the ſphere, repreſented by the intercepted arch on the plane of projection, is the meaſure of the repreſented ſpheric angle (1. Cor. Prop. VI. B. II.), and therefore of the projected angle which is equal to it.

PROP. VIII. THEOREM.

THE centre of the projection of a great circle inclined to the primitive, is in the line of meaſures, at a diſtance from the centre of the primitive equal to the tangent of the circle's inclination to the primitive; and the ſemidiameter of the projection is equal to the ſecant of the circle's inclination.

Fig. 54. Let ABIH be the projection of a great circle on the plane of the primitive AEID; I ſay the diſtance of their centres is equal to the tangent, and the ſemidiameter of the projection ABIH is equal to the ſecant of their inclination to each other.

Through C, O their centres draw the line of meaſures

CO (1. Cor. Prop. II.), and A, I being the interſections of the projection with the primitive, join AI, AO. Becauſe AI paſſes through the centre C of the primitive (Prop. III.), therefore from the centres C, O are drawn CA, OA to one of the points A in which the circles meet each other, and conſequently the angle CAO meaſures the inclination, which the great circle repreſented by ABI has to the primitive (Cor. Prop. XII.). But AI is perpendicular to CO (Prop. III.), wherefore a circle being deſcribed about A as a centre with the diſtance AC the ſemidiameter of the ſphere, CO will be the tangent, and AO the ſecant of the angle CAO, viz. of an angle which is the meaſure of the inclination of the great circle to the primitive.

Cor. If AO meet the primitive in L, and CL be joined, the intercepted arch IL, and the angle ICL will each be double to the meaſure of the inclination of the great circle to the primitive.

For the angle ICL is double to the angle IAL (20. E. 3.), and the angle ICL is meaſured by the arch IL (4. Def. B. III.).

PROP. IX. THEOREM.

THE centre of the projection of a leſs circle perpendicular to the primitive is in the line of meaſures at a diſtance from the centre of the primitive equal to the ſecant of the circle's diſtance from its nearer pole, and the radius of the projection is the tangent of the ſame diſtance.

Fig. 55. LET RAPI be the primitive, C its centre, bHhB the projection of a circle perpendicular to the primitive, cutting it in the right line bh, O the centre of the projection, and CO the line of meaſures meeting the primitive in P, R, the projection in B, H. P, R are the poles of the circle (5. COR. DEF. B. IV.), of which P is the pole nearer to the circle; I ſay therefore that CO is equal to the ſecant of the circle's diſtance from the pole P, and Ob or Oh is equal to the tangent of the ſame diſtance.

DRAW ACI perpendicular to the line of meaſures, the points, A, B, b, and A, H, h are in a ſtraight line (3. COR. PROP. V.). Wherefore the angles Abh, OHh are equal to each other, becauſe ſtanding upon the ſame arch Bh of the circle BbH, but becauſe Oh is equal to OH, the angle OHh is equal to the angle OhH, therefore the angle OhH is equal to the angle Abh. Wherefore becauſe the right line Oh meets the circle ARI in h, but Ah drawn through the ſame point cuts it, and the angles which Oh makes with Ah are equal to thoſe in the alternate ſegments of the circle into which Ah divides it, Oh ſhall touch the circle (CONVERSE. 32. E. 3.). Therefore CO is the ſecant, and Oh, which is the ſemidiameter of the projection, is the tangent of the arch Ph, viz, of the diſtance of the circle from its nearer pole P.

COR. IF a leſs circle be perpendicular to the primitive, right lines drawn from the centres of the primitive and of the projection to either concourſe of the projection with the primitive, will mutually touch each circle.

FOR becauſe Oh touches the primitive in h, Ch is perpendicular to hO, and hO paſſes through the centre O of the projection, therefore Ch alſo touches the projection (COR. 16. E. 3.). And the demonſtration of the Propoſition would have been the ſame, if Ob had been drawn inſtead of Oh.

PROP.

PROP. X. THEOREM.

THE poles of a circle of the ſphere meeting the primitive, are projected in the line of meaſures at diſtances from the centre of the primitive equal to the tangent and cotangent of half the circle's inclination to the primitive.

CASE 1. WHEN the circle is perpendicular to the plane of the primitive.

THE poles of every circle perpendicular to the primitive, are in the periphery of the primitive (5. COR. DEF. B. II.). Wherefore the poles of a circle, whoſe inclination to the primitive is that of a right angle, are each projected at a diſtance from the centre of the primitive equal to the ſemidiameter of the primitive, which is both the tangent and cotangent of half a right angle.

CASE 2. WHEN the circle is inclined obliquely to the primitive.

Fig. 54. 56. BECAUSE the poles of every circle of the ſphere are in the plane of the right circle making the line of meaſures (8. DEF. B. IV.), they will be projected in the line of meaſures. Wherefore let p, r be the projected poles of a circle of the ſphere inclined to the primitive, and ſituated in the line of meaſures ED. Draw ACI a diameter of the primitive at right angles to ED, and join Ap, Ar meeting the primitive in P, R, alſo join PR. The arches IP, IR meaſure the circle's inclination to the primitive, and the whole circumference PIR is a ſemicircle (5. COR. PROP. V.). Wherefore PAR or pAr is a right angle, and ICP is the acute angle meaſuring the circle's inclination. But the angle ICP at the centre is double to the angle IAP at the circumference (20. E. 3.), therefore the angle pAC is the meaſure of half the circle's inclination, and the angle CAR is the compliment of the ſame; and a circle being deſcribed about the angle A as a centre with the diſtance of AC the radius of the ſphere, Cp, Cr are the tangents of the angles CAp, CAr. Wherefore the projected poles p, r are diſtant from the centre of the primitive in the line of meaſures, at intervals equal to the tangent and cotangent of half the circle's inclination to the primitive.

COR. 1. THE poles of every circle of the ſphere which is not parallel, nor perpendicular to the primitive, are projected the one within, the other without the primitive and the projection, but towards different parts of the centre, viz. that which is projected within the primitive, is the pole which is nearer to the pole of the primitive oppoſite to the projecting point.

For becauſe the arch IP is leſs, and the arch IR greater than a quadrant, the pole P which is nearer to I is between the terms I, D, and the pole R external to the terms I, E. Wherefore the right lines AP, AR muſt meet the line of meaſures ED on different ſides of the centre, the one between the centre C and the extremity D of the diameter ED, the other towards the parts of the centre external to the other extremity E.

In the ſame manner becauſe the circumference bPh is biſected in P, and the remaining circumference bRh is biſected in R, therefore the point P is within the terms b, h, but the point R is without theſe terms; therefore the right lines AP, AR fall the one within, the other without the angle bAh, viz. the points p, r fall the one within, the other without the projection BGH.

Cor. 2. If a diameter of the primitive be drawn at right angles to the line of meaſures of any circle of the ſphere inclined to the primitive, and from either extremity of this diameter right lines be drawn to its projected poles, the angle comprehended by theſe right lines will be a right angle, and the angle comprehended by the diameter and the right line drawn to the internal pole will meaſure half the circle's inclination to the primitive.

PROP. XI. THEOREM.

THE line of meaſures of every circle of the ſphere, whoſe plane is not parallel to the primitive, is harmonically divided in the projected poles, and in the points of concourſe with the projection.

Let BGH be the projection of a circle of the ſphere, whoſe plane is not parallel to the primitive, ED the line of meaſures paſſing through the projected poles p, r, and meeting the projection in B, H; I ſay, the line of meaſures is harmonically divided in the points p, r, B, H. *Fig.* 54. 56.

If the projection be that of a great circle, its interſections A, I with the primitive are in a diameter to the primitive at right angles to the line of meaſures ED (Prop. III.), but if the projection be that of a leſs circle, draw the diameter ACI of the primitive at right angles to ED, and join AB, AH, Ap, Ar, meeting the primitive in b, h, P, R. The circumferences bPh, bRh are biſected in P, R (4. Cor.

Prop. V.), and the angle pAr is a right angle (2. Cor. Prop. X.). Produce rA to K, becauſe pA is perpendicular to rA, the angle pAK is equal to the angle rAp, and becauſe of the equal arches Ph, Pb, the angle pAH is equal to the angle pAB (27. E. 3.), wherefore the remaining angle HAK is equal to the remaining angle BAr.

The right line rAK is therefore drawn externally to make equal angles with AB, AH the ſides of the triangle BAH, and meets the baſe BH in r, as Ap meeting the baſe in p makes equal angles internally with the ſame ſides. Wherefore the baſe is divided in the points r, p in the proportion of the ſides (3. E. 6. Simſon's Edition.), viz. as pB is to pH ſo is BA to AH and as rB is to rH ſo is BA to AH. Therefore, ex æquo, pB is to pH as rB is to rH, which is the harmonic ſection of the right line pr.

Cor. The line of meaſures of a leſs circle perpendicular to the primitive is harmonically divided in its concourſe with the primitive and with the projection.

For the concourſes with the primitive are the projected poles (Cor. 2. Prop. V.).

PROP.

PROP. XII. THEOREM.

THE diſtances of either projected pole of a great circle, inclined to the primitive, from the centres of the primitive and of the projection, are proportional to the ſemidiameter of the primitive and of the projection.

LET ABIH be the projection of a great circle cutting the primitive in A, I, and its line of meaſures in B, H; and let C, O be the centres of the primitive and of the projection, p, r, the projected poles of the great circle, each in the line of meaſures ED. I ſay pC, pO and rC, rO are proportional to the ſemidiameters of the primitive and of the projection ABIH. *Fig.* 54.

DRAW the diameter ACI (PROP. III.), and join AO, Ap, Ar, producing Ar to K. The angle CAO is the meaſure of the great circle's inclination to the primitive (1. COR. PROP. VII.), and the angle CAp is the meaſure of half the ſame inclination (2. COR. PROP. X.), wherefore the right line Ap biſects the angle CAO, and becauſe Ap, Ar are at right angles to each other (2. COR. PROP. X.), the angle rAp is equal to the angle pAK, and the parts CAp, pAO having been ſhewn to be equal, the remaining angles rAC, OAK are equal between themſelves. Wherefore Ap makes equal angles internally, and Ar externally, with the ſides AC, AO of the triangle CAO. Conſequently pC is to pO, and rC is to rO, as AC is to AO (3. E. 6. Simſon's Edition.), viz. as the ſemidiameter of the primitive is to the ſemidiameter of the projection ABIH.

COR. THE line of meaſures of every great circle inclined to the primitive is harmonically divided in the projected poles, and in the centres of the primitive and projection.

FOR, becauſe pC is to pO as AC to AO and alſo rC is to rO as AC to AO, therefore, ex æquo, pC is to pO as rC is to rO.

L E M. XI.

IF through a point within a circle a right line be drawn to meet the circumference, the angle which it makes with the ſemidiameter drawn to the point of concourſe ſhall be leſs than a right angle.

BUT if the point be without the circle, as the right line meets the circumference in the concave or convex circumference, the angle which it makes with the ſemidiameter drawn to the point of concourſe ſhall accordingly be leſs or greater than a right angle.

Let BEF be a circle, C its centre, and A any point within it, through which are inflected AB, CB to the circumference; I ſay the angle ABC is leſs than a right angle. Becauſe the point A is within the circle, the right line AB will meet it in another point b towards the oppoſite parts of A. Join Cb, the angles CBb, CbB are equal between themſelves (5. E. 1.), and therefore each leſs than a right angle, becauſe the three angles are together only equal to two right angles (32. E. 1.). Wherefore the angle ABC is leſs than a right angle. *Fig.* 57.

But if D be a point without the circle, from which a right line DbB is drawn, DB falling upon the concave, and Db upon the convex circumference, and the ſemidiameters CB, Cb be drawn, the angle DBC ſhall be leſs, but the angle DbC greater than a right angle. Becauſe the concave circumference is remoter from D, the point b is between the terms D, B, wherefore DBC is the ſame with the angle bBC, viz. is leſs than a right angle; and becauſe the angle BbC is alſo leſs, but the angles BbC, DbC together are equal to two right angles, the angle DbC is alone greater than a right angle.

D E F.

DEF.

IF two circles cut each other, and through a point within each of them a right line be drawn to meet each circumference towards different parts of the point, the right line is ſaid to meet them towards LIKE PARTS; but if it fall on each circumference towards the ſame parts, then is it ſaid to meet them towards UNLIKE PARTS.

AND if a right line be drawn from a point without each circle to meet each circumference in the concave or convex parts, it is ſaid to meet each towards LIKE PARTS; but if it fall on the one on the concave, the other on the convex circumference, then is it ſaid to meet each circumference towards UNLIKE PARTS.

PROP. XIII. THEOREM.

IF through the projected pole of a great circle and the extremities of any arch of the projection right lines be drawn to meet the primitive and projection towards UNLIKE PARTS, the arch of the primitive intercepted thereby, will meaſure the arch of the projection.

Fig. 20. CASE I. WHEN the great circle is a right one.

LET AEID be the primitive, whoſe centre is C, and the diameter ECD the rectilinear projection of a great circle paſſing through the poles of the primitive of which bp is any portion. Draw ACI perpendicular to ED, then A, I are the projected poles of the projection ECD (5. Cor. Def. B. IV.). Draw Ab, Ap meeting the primitive in B, P, the intercepted arch BP of the primitive is by the ſame reaſoning as in Prop. V. ſhewn to be equal to the arch repreſented by bp.

Case 2. When the projection is of a great circle inclined obliquely to the primitive, the projected pole the internal one, and one of the right lines drawn is the line of meaſures of the great circle.

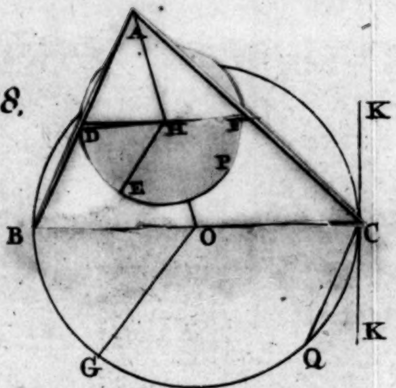
Let aBi be a great circle of the ſphere, abi its projection on the plane of the primitive EaD, whoſe pole A is the projecting point, and P that pole of the circle aBi whoſe projection is the internal pole p. If a right circle EPD paſſing through the pole P cut the great circle and its projection in the points B, b towards the ſame parts of the pole p, and the primitive in the line of meaſures EpD, and through any other point f in the projection abi be drawn the right line pf meeting the primitive in K towards the ſame parts with f; I ſay the arch EK ſhall be equal to the arch of which bf is the repreſentative. *Fig.* 29.

Let BH, ai be the common ſections of the great circle aBi with the primitive and the right circle EPD, cutting each other in the centre C of the ſphere, becauſe A, B, b are in a ſtraight line, as alſo A, P, p, and A, F, f (I. DEF. B. IV.), let theſe ſtraight lines be drawn, and let pf, AP meet ai, BH in L, G. Becauſe the points f, p are in the plane AFP, the right line pLf will be the common ſection of the plane AFP with the primitive, and therefore the three points F, L, G are in the plane AFP, and they are alſo in the plane aBi, wherefore they are in the common ſection of the planes AFP, aBi, viz. FLG is a ſtraight line. And becauſe P, A are the poles of the circles aBi, aEi, the arches PH, AD are equal, being each a quadrant (1. COR. PROP. IV. B. II.), wherefore taking away the common arch HD, the remaining arches PD, AH, and conſequently the angles PCp, ACG, are equal between themſelves. But becauſe CP is equal to CA, the angle CPp is equal to the angle CAG, therefore the triangles CpP, CGA are equiangular, and CP being equal to CA, the homologous ſides Cp, CG are alſo equal (26. E. 1.). Again, becauſe the planes EaD, BaD are each perpendicular to the plane EPD, their common ſection aC is perpendicular to EPD (19. E. 11.), and conſequently to the right lines Cp, CG, wherefore the angles LCp, LCG are each a right one, and the ſides LC, Cp, being equal to the ſides LC, CG, each to each, the angles CpL, CGL are equal between themſelves (4. E. 1.). Therefore of the triangles

KCp,

E
F

18.



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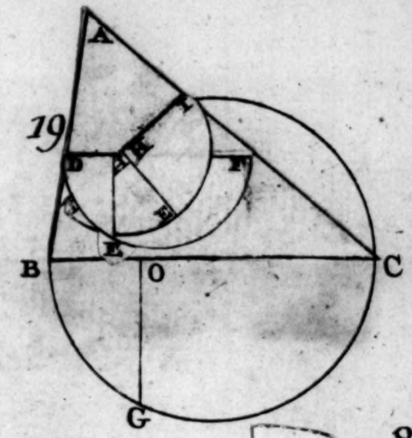

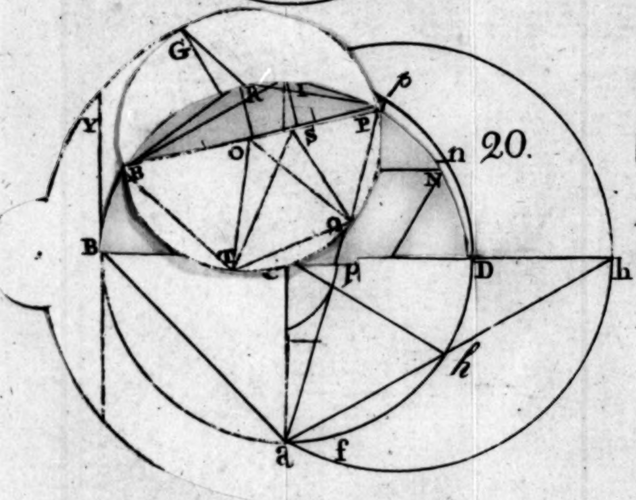

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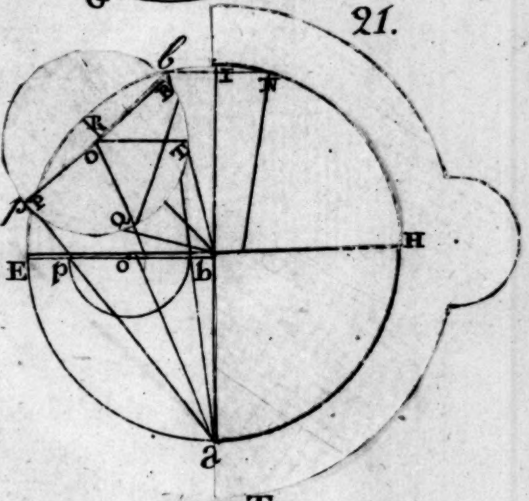
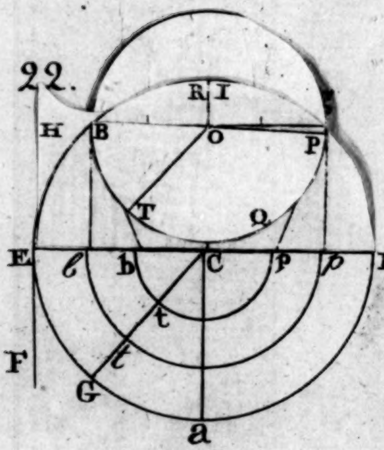
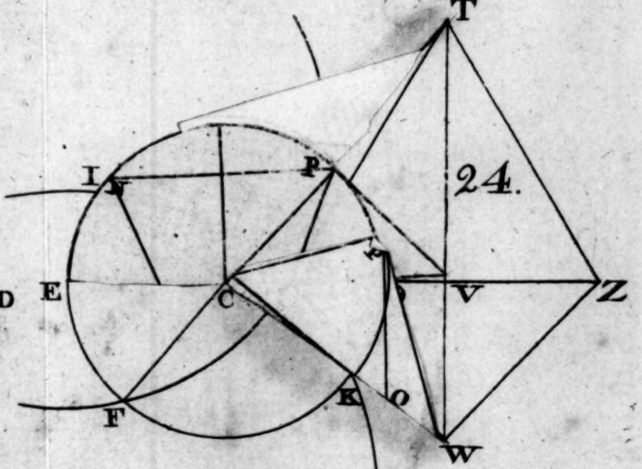






TAB. II. *pag.* 200.

KCp, FCG, the ſide KC being equal to the ſide FC, and Cp to CG, and of the homologous angles, the two CpK, CGF are equal, but the angles LCp, LCG being right, the whole angles KCp, FCG are greater than right ones, and conſequently the remaining angles CKp, CFG are each leſs than right, therefore the triangles are equiangular (Lem. I.), viz. the homologous angles CKp, CFG are equal between themſelves. Wherefore the ahgles CKp, KpC being equal to the angles CFG, CGF, the external angles KCE, FCB (32. E. 1.), and conſequently the arches KE, FB are equal between themſelves (26. E. 3.). But the arch FB is that of which fb is the repreſentative, therefore the arch KE is equal to the arch repreſented by fb, or it meaſures the arch fb of the projected great circle abi.

Case 3. Let fq be any arch of the projected great circle abi, and pf, pq be drawn meeting the primitive in K, O; I ſay the arch KO is equal to that repreſented by fq.

For the ſame things remaining, if FQ be the arch repreſented by fq, becauſe EK is equal to BF, and EO to BQ, the whole or remaining arch KO is equal to the whole or remaining arch FQ, viz. to the arch repreſented by fq.

And in the ſame manner is the Propoſition demonſtrated, if the right lines be drawn through the external projected pole.

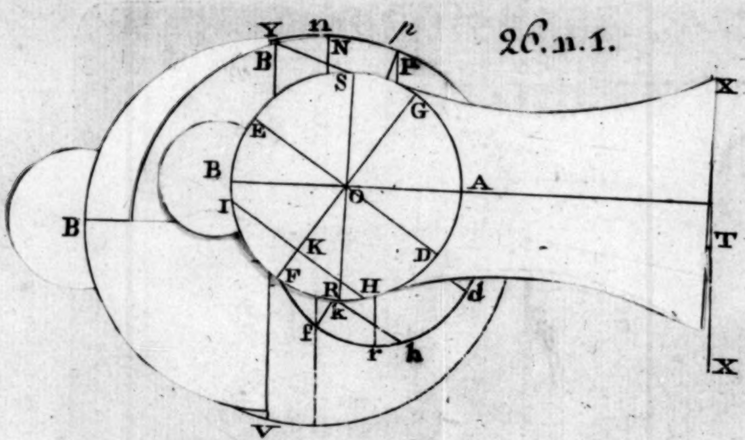
COR. IF from the angular point of a projected ſpheric angle two right lines be drawn through the projected poles of the ſides; the intercepted arch of the primitive will meaſure the ſpheric angle.

Fig. 32. LET ABC be a projected ſpheric angle, comprehended by the great circles AB, BC, whoſe poles are c, *a*. Deſcribe the great circle c*a*, which paſſing through the poles of AB, BC is perpendicular to each of them (PROP. III. B. II.). Wherefore AB, BC do alſo each paſs through the poles of c*a*, and conſequently B is one of its poles; therefore right lines drawn from B through c, *a* will intercept an arch of the primitive equal to the arch repreſented by c*a*. But the arch repreſented by c*a* is the meaſure of the ſpheric angle repreſented by ABC (PROP. VI. B. II.), viz. of the projected angle ABC itſelf (PROP. VII.), therefore the arch of the primitive which would be intercepted by the right lines Bc, B*a*, meaſures the angle ABC.

PROP. XIV. THEOREM.

IF through the projected pole of a great circle and the extremities of an arch of the projection, right lines be drawn to meet the primitive and

26.n.1.



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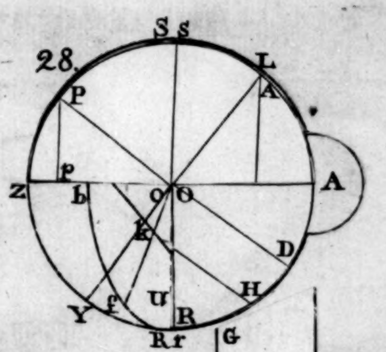

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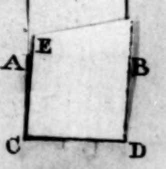

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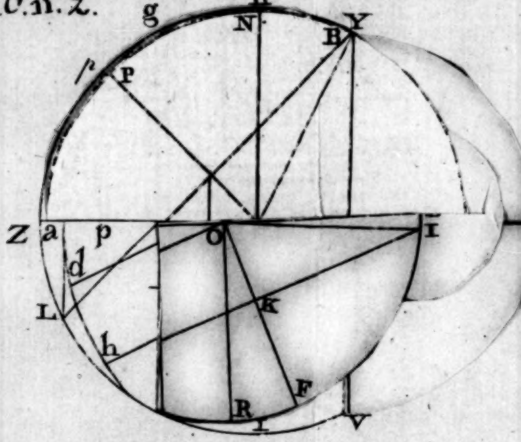

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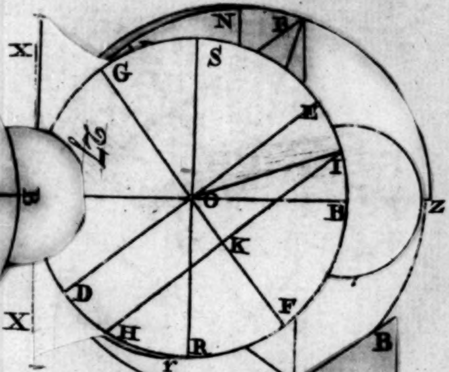

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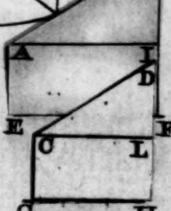

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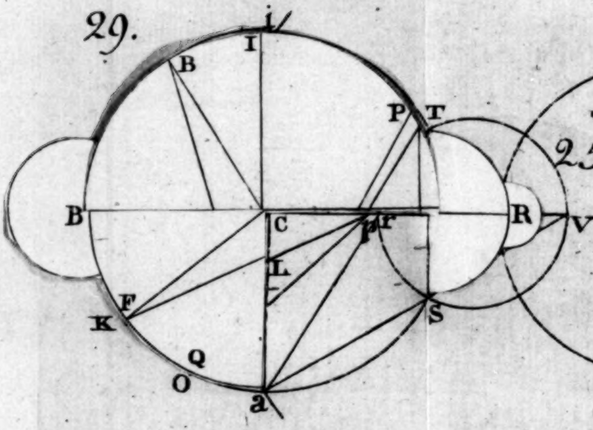

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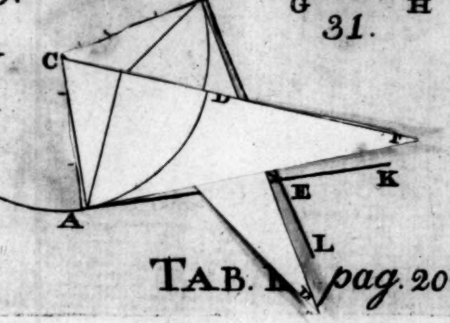


TAB. I. pag. 202.

and projection towards like parts, the intercepted arches of the primitive and of the projection ſhall be ſimilar.

Case i. When one of the right lines drawn is the line of meaſures.

Let ABIH be the projection of a great circle, AEID the primitive, CO the line of meaſures paſſing through the centres C, O of the primitive and of the projection, and meeting them in E, D, B, H; P the internal or external projected pole, through which is drawn the right line KPG meeting the primitive in K, the projection in G (towards different parts of the internal pole, but in the caſe of the external pole, to meet the primitive and the projection each in the concave, or each in the convex circumference), then E, H being the interſections of the line of meaſures with the primitive and projection towards the ſame parts with K, G; I ſay, the arches DK, BG, and EK, HG ſhall be ſimilar portions of their reſpective circumferences. *Fig.* 59. 60.

Join CK, OG. The angles CKP, OGP are each leſs, or each greater than a right angle (Lem. XI.). Wherefore in the triangles CKP, OGP, two angles, one in each, are either both leſs, or both greater than right angles, two other angles at P are either one and the ſame, or oppoſite and equal angles, and the ſides about the remaining angles at C, O are proportional, viz. PC is to PO as CK to OG (PROP. XII.). Therefore the triangles are equiangular, and the homologous angles PCK, POG are equal (7. E. 6.). Conſequently the external angles ECK, HOG are alſo equal. But equal angles at the centres of circles are ſubtended by ſimilar circumferences (LEM. VIII.). Therefore the circumferences DK, BG, and EK, HG are ſimilar.

CASE 2. LET GQ be any arch of the projected great circle ABI, and PG, PQ be drawn meeting the primitive in K, F, towards like parts with G, Q. The arches GQ, KF ſhall be ſimilar portions of their reſpective circumferences.

FOR the arches HG, EK, and HQ, EF are ſimilar, viz. they are proportional to the whole circumferences of which they are parts; and therefore (dividendo or compendo) the arches GQ, KF ſhall alſo be ſimilar portions of their reſpective circumferences.

COR. LET PG, PQ meet the primitive and the projection again in the points k, f, and g, q. The arches GQ, gq ſhall mutually be the meaſures of each other, viz. GQ ſhall be the ſame part of the whole projected circumference

ABI, as the arch repreſented by gq is of the whole circumference of the great circle repreſented by ABI.

For the arch FK meaſures the arch gq, viz. it is the ſame portion of the primitive, that the arch repreſented by gq is of the great circle repreſented by AB (Prop. XIII.), but the arch GQ is ſimilar to the arch KF, therefore the arch GQ is the ſame portion of the projection ABI, that the arch repreſented by gq is of the great circle repreſented by ABI.

And in the ſame ſenſe, and for the ſame reaſon, is the arch gq the meaſure of GQ.

PROP. XV. THEOREM.

If the projection of a great circle paſs through one interſection of the projection of another great circle with its line of meaſures, it will paſs through the other alſo.

Let ABIH be the projection of a great circle, whoſe line of meaſures is BH meeting the projection in B, H; if the projection DBE of another great circle paſs through the point B, it will paſs through the point H alſo. *Fig.* 58.

FOR BH being the projection of a diameter of the great circle of which ABI is the repreſentative, the points from which B, H are projected are diametrically oppoſite on the ſurface of the ſphere. But ſince the common ſection of great circles is a diameter of the ſphere (PROP. II. B. II.), the great circle repreſented by DBE as it paſſes through the point correſponding to B, muſt paſs through the point correſponding to H alſo. Wherefore B, H, being the projections of two points on the ſurface of the ſphere which are common to the two great circles, are common to both the projections DBE, ABI.

COR. 1. HENCE every circle, paſſing through the interſections of a great circle with its line of meaſures, is itſelf the repreſentative of a great circle.

COR. 2. THE LOCUS of the centres of the projections of great circles paſſing through the ſame point on the plane of the primitive, is a right line; and if the point be given, the Locus itſelf is given.

FOR through O the centre of the projection ABI draw OF at right angles to BH; becauſe BH is biſected in O, the centre of every circle paſſing through the points B, H is in the right line OF (COR. 1. E. 3.). Wherefore the

Locus of the centres of all the projections of great circles paſſing through the point B, is a right line.

And if the point B be given, this right line is given in poſition. For the centre of the primitive being alſo given, the line of meaſures BC is given in poſition; but the projection of every great circle cutting the primitive in a diameter to the primitive at right angles to the line of meaſures (Prop. III.); this diameter, viz. ACI is given in poſition and magnitude. Wherefore the three points A, B, I, through which the projection ABI paſſes, being given, the circle itſelf is given, and therefore the point H. Wherefore BH being given in poſition and magnitude, the right line OF biſecting it at right angles is given in poſition.

Cor. 3. Hence every circle paſſing through either of the points B, H, and whoſe centre is in the right line OF, is the repreſentative of a great circle.

PROP. XVI. PROBLEM.

THE ſtereographic projection of a circle of the ſphere being given, to find the projection of its pole.

CASE 1. IF the given projection be parallel to the primitive, the projected pole is in the centre of the primitive (6. COR. DEF. B. IV.).

CASE 2. IF the given projection be that of a right circle, or of a leſs circle perpendicular to the primitive, the extremities of a diameter of the primitive at right angles to the rectilinear projection, or to the right line joining the points in which the projection cuts the primitive, will be the projected poles (5. COR. DEF. B. IV.).

Fig. 54. 56. CASE 3. LET BGH be the circular projection of any circle of the ſphere, whoſe projected pole is required.

THROUGH C the centre of the primitive and the centre of the projection draw the line of meaſures cutting the projection in B, H. If the projection be that of a great circle, its interſections A, I with the primitive are in a diameter of the primitive at right angles to BH; but if it be that of a leſs circle, draw the diameter ACI of the primitive at right angles to BH, and join AB, AH meeting the primitive in b, h. Biſect the circumferences bIh, bAh in P, R, and join AP, AR meeting BH in p, r. The points p, r are the projected poles (4. COR. PROP. V.), of which p is the internal, and r the external pole (1. COR. PROP. X.). Or, becauſe the arch IP meaſures the circle's inclination to the primitive (4. Cor. Prop. V.), and Cp, Cr are equal to the tangent and cotangent of half the arch IP, the points p, r will be the projected poles (Prop. X.).

Cor. 1. If the inclination of the circle to the primitive be required, the arch IP, the meaſure of the inclination, is found by this Propoſition.

Cor. 2. If the inclination be given, draw the diameter ACI as above, and making IP equal to the meaſure of this inclination, or drawing AP making with AI the angle PAI equal to half this inclination, the right line AP ſhall meet BH in p the internal projected pole.

Schol. If the projection be that of a great circle, the right line ABb being drawn as before, and E being the adjoining interſection of the line of meaſures with the primitive, make the arches IP, AR each equal to the arch Eb, and draw ApP, ARr. The points p, r ſhall be the projected poles. Draw AhH, becauſe IP is equal to Eb, add to each the common arch Ib, and the whole Pb is equal to the whole EI, viz. to a quadrant. But the arch bIh is a ſemicircle, therefore it is double to the arch Pb, and conſequently is biſected in P. Wherefore as before p is one projected pole of the circle repreſented by ABI, and for the ſame reaſon is r the other projected pole. *Fig.* 54.

PROP. XVII. PROBLEM.

ABOUT a given point in the plane of the primitive, as a projected pole, to deſcribe the projection of a circle, whoſe diſtance from its pole ſhall be equal to a given arch of the primitive.

CASE I. WHEN the given point is in the centre of the primitive.

Fig. 56. LET IP be the given arch of the primitive, draw the diameter ICA of the primitive, and DCE at right angles to it; join AP meeting CD in p, and the circle deſcribed round the centre C with the diſtance Cp ſhall be the projection required.

FOR the projected pole being in the centre of the primitive, the circle to be repreſented by the projection is parallel to the primitive (CONVERSE. 6. COR. DEF. B. IV.), and therefore has the ſame poles with the primitive. Wherefore the centre of the projection is the centre of the primitive itſelf, and the ſemidiameter of the projection is equal to the ſemitangent of the circle's diſtance from the oppoſite pole of the primitive (2. Cor. Prop. II.), viz. from one of its own poles, or of the given arch IP. But Cp is the ſemitangent of IP, therefore the circle deſcribed round C with the diſtance Cp is the projection required.

Case 2. When the projection is that of a great circle, and the given point is in the circumference of the primitive.

Let D be the given point in the circumference of the primitive AEID, draw the diameter DCE of the primitive, and ACI at right angles to it. AI ſhall be the projection required. For DE is the line of meaſures of the circle (Prop. X.), and becauſe the projected pole is in the circumference of the primitive, the projection is that of a circle perpendicular to the plane of the primitive (Converse. 5. Cor. Def. B. IV.). But the circle being alſo a great one, it paſſes through the poles of the primitive (Prop. III. B. II.), wherefore the projection is a diameter of the primitive at right angles to the line of meaſures (Prop. I. and Cor.), viz. ACI is the projection required.

Case. 3. When the given point is neither in the centre, nor in the circumference of the primitive.

Let p be the given point in the plane of the primitive AEID. Through C the centre of the primitive and p draw the *Fig.* 54. 56.

the right line Cp, which will be the line of meaſures (PROP. X.), and draw the diameter ACI of the primitive at right angles to Cp, alſo join Ap meeting the primitive in P. On each ſide of P make the arches Pb, Ph equal to the given arch of the primitive, and join Ab, Ah cutting the line of meaſures in B, H. A circle BGH whoſe diameter is BH ſhall be the projection required.

BECAUSE ACI is a diameter of the primitive perpendicular to the line of meaſures Cp, and from one extremity A thereof are drawn right lines through the points B, H in which the line of meaſures meets the projection BGH, the intercepted arch bih of the primitive ſhall be double to the diſtance of the circle repreſented by BGH, from its pole, and this arch being biſected in P, the point p ſhall be the projection of one of its poles (4. COR. PROP. V.). Wherefore the circle of the ſphere repreſented by BGH is at a diſtance from its pole equal to the given arch of the primitive, and its projected pole is the given point p.

CASE 4. WHEN the given point is in the circumference of the primitive, but the projection is that of a leſs circle.

Fig. 55. LET P be the given point, towards each part of which make the arches Pb, Ph equal to the given arch of the primitive, draw the line of meaſures CP, and ACI through the centre of the primitive at right angles to CP, and join Ab, Ah meeting CP in B, H. BH ſhall be the diameter of the projection required.

As the conſtruction is the ſame, ſo is the demonſtration the ſame with that in CASE 3.

COR. IN CASE 1., viz. where the projection is that of a circle parallel to the primitive, as the given arch of the primitive is leſs or greater than a quadrant, the projection will fall within, or without the primitive, and will repreſent a circle of the ſphere remoter from the projecting point than the primitive, or nearer to it. Therefore if the projection be required of a circle parallel to the primitive, but remoter from the projecting point, the given arch of the primitive muſt be leſs than a quadrant, being the circle's diſtance from its nearer pole; and if the circle to be repreſented be nearer to the primitive, the given arch of the primitive muſt be its diſtance from its remoter pole, and therefore greater than a quadrant.

SCHOL. 1. IF the projection be required to be of a great circle, becauſe the arch IP is given, the tangent and ſecant of the ſame are given. But IP meaſures the inclination of the circle (5. COR. PROP. V.), wherefore make CO equal to the tangent of IP, and O ſhall be the centre of the projection required (PROP. VIII.), whoſe ſemidiameter is the ſecant of the inclination, viz. is OA or OI.

SCHOL. 2. IF the projection be that of a leſs circle perpendicular to the primitive, the centre of the projection is at a diſtance from the centre of the primitive equal to the ſecant, and the ſemidiameter of the projection is equal to the tangent of the circle's diſtance from its nearer pole (PROP. IX.). But this diſtance is given, therefore the ſecant and tangent of it are given, viz. the centre and the ſemidiameter of the projection are given.

PROP. XVIII. PROBLEM.

THROUGH two given points on the plane of the primitive to deſcribe the projection of a great circle. But it is neceſſary that the points be not both in the circumference of the primitive.

FOR if the given points be both in the circumference of the primitive, they muſt be in the ſame right line with the centre of the primitive, becauſe the interſections of every projected great circle with the primitive are in a diameter to the primitive (Prop. III.). But this diameter itſelf, and every circle paſſing through theſe interſections will be the repreſentative of a great circle, and therefore the Problem would be unlimited.

Case 1. When one of the given points is in the circumference of the primitive.

Let AEID be the primitive, C its centre, and G, I two points in the plane of the primitive, of which I is in the circumference. Draw the diameter ICA and deſcribe a circle through the points A, G, I, this ſhall be the projection required. For it cuts the primitive in two points A, I which are the extremities of a diameter to the primitive, and therefore AGI is the repreſentative of a great circle (Prop. III.). *Fig.* 58.

Case 2. When neither of the given points are in the circumference of the primitive.

Let B, K be the two given points, through either of which, as B, draw the diameter BC of the primitive, and ACI at right angles to it. Through the points A, B, I deſcribe a circle, which, by Case 1., will repreſent a great circle. Let it meet BC in H and deſcribe a circle through B, K, H; this ſhall be the projection required. For BH, being perpendicular to AI is the line of meaſures of the projected great circle ABI (PROP. III.); and B, H are the points in which this line of meaſures meets the projection, therefore every circle in the plane of the primitive paſſing through B, H is the repreſentative of a great circle (1. COR. PROP. XV.), viz. the circle BKH repreſents a great circle, and it is deſcribed through the given points B, K.

Or, CASE 2. may be thus demonſtrated.

THE ſame conſtruction remaining, let the projection BKH meet the primitive in D, and draw DC meeting the projection in E. Becauſe the right lines BH, AI within the circle ABI meet each other in C, the rectangle BCH is equal to the rectangle ACI (35. E. 3.). For the ſame reaſon in the circle BKH is the rectangle BCH equal to the rectangle DCE, therefore the rectangle DCE is equal to the rectangle ACI; and becauſe DC is equal to AC, CE ſhall be equal to CI, and conſequently the point E is in the circumference of the primitive. Wherefore the circle BKH cutting the primitive in two points which are in a diameter of the primitive, is the repreſentative of a great circle (PROP. III.).

PROP.

PROP. XIX. PROBLEM.

TO meaſure on the circumference of the primitive a given portion of the projection of a circle. And converſely, from a given projection and at a given point therein to cut off a portion which ſhall repreſent an arch equal to a given arch of the primitive.

Case I. When the projection is of a right circle.

Let the diameter ECD be the projection of a right circle, and Bp any portion thereof. Draw the diameter ACI of the primitive at right angles to ED, and from either extremity A draw AB, Ap meeting the primitive in b, P, the arch bP ſhall meaſure the portion Bp. For A is the projected pole of the great circle repreſented by ED, and therefore the arch bP of the primitive is epual to the arch repreſented by the portion Bp (Prop. XIII.). *Fig.* 54.

And converſely, if it be required at the point B and in the projection ED to cut off a portion repreſenting an arch equal to a given arch of the primitive; the ſame things remaining, draw ABb, and make the arch bP equal to the given arch of the primitive. Join AP meeting ED in p, and Bp ſhall be the portion required.

CASE 2. WHEN the projection is of a great circle inclined obliquely to the primitive.

Fig. 59. 60. LET ABI be the circular projection of a great circle on the plane of the primitive AEID, and GQ a portion thereof. It is required to find an arch of the primitive equal to the arch repreſented by GQ. Find P one of the poles of the projection ABI (PROP. XVI.), and draw PG, PQ, meeting the primitive in unlike parts k, f; the arch kf is the meaſure of the projected arch GQ (PROP. XIII.).

CONVERSELY, IF at the point G in the given projection ABI it be required to cut off a portion equivalent to a given arch of the primitive; the ſame things remaining, draw PG meeting the primitive in k, and making kf equal to the given arch of the primitive, join Pq meeting the projection in Q. GQ ſhall be a portion of the projection ABI repreſenting an arch equal to kf (PROP. XIII.).

CASE 3. WHEN the projected circle is the repreſentative of a leſs one.

LET FGK be the projection of a leſs circle, and FG a portion thereof, which it is required to meaſure on the circumference of the primitive. Find P either projected pole of FGK, round which deſcribe a great circle ABI (PROP. XVII.), and through P, F, and P, G deſcribe two great circles PF, PG (PROP. XVIII.) meeting the projection ABI in B, H. Draw the right lines PB, PH meeting the primitive in unlike parts b, h, and the arch bh ſhall meaſure the portion FG of the projection FGK. *Fig.* 61. 62.

FOR the circles ABI, FGK having the ſame projected pole, repreſent two parallel circles of the ſphere, and the projections PB, PH repreſent two great circles deſcribed through their common pole. Wherefore the intercepted portions BH, FG repreſent ſimilar arches of the parallel circles (PROP. XVII. B. II.). But the arch bh of the primitive is equal to the arch repreſented by BH, and therefore is ſimilar to the arch repreſented by FG, the given portion of the projected leſs circle FGK.

CONVERSELY, If at the point F it be required to cut off from the given projection FGK a portion ſimilar to a given arch of the primitive; the ſame things remaining, draw the great circle PFB, and the right line PBb; on the primitive make the arch bh equal to the given arch, and draw the right line Ph meeting the great projection ABI in H;

through P, H deſcribe the great circle PH meeting the leſs projection in G; the arch FG ſhall repreſent an arch ſimilar to the given arch of the primitive.

Fig. 63. IF the projection be of a leſs circle parallel to the primitive, becauſe the pole P and the centre C of the primitive coincide, the great circle ABI becomes the primitive itſelf, and the great circles PF, PG being the projections of right circles, become diameters of the primitive, alſo the points B, b, and H, h coincide. Wherefore the arch FG of the parallel projection FGK being given, draw through the centre of the primitive CFB, CGH, and the intercepted arch BH of the primitive ſhall be the meaſure of FG. Alſo converſely, if at the point F it be required to cut off from the projection FGK a portion ſimilar to a given arch of the primitive; draw CFB, and on the circumference of the primitive make BH equal to the given arch of the primitive, draw CH meeting the projection in G, and FG ſhall be the portion required.

SCHOL. to CASE 2. AN arch of the projection of an oblique great circle may be meaſured on the circumference of the projection itſelf.

59. 60. FOR the ſame things remaining, let the right lines PG, PQ meet the projection again in g, q, the arches GQ, gq ſhall be mutually the meaſures of each other (COR. PROP. XIV.),

viz. the arch GQ ſhall be a ſimilar portion of the circumference ABIH, as the arch repreſented by gq is of the great circle repreſented by the projection ABI, or if OG, OQ be joined, the angle GOQ ſhall be equal to the angle ſubtended at the centre of the ſphere by the arch repreſented by the arch gq. And the ſame meaſure ſhall the arch gq be of the arch GQ.

PROP. XX. PROBLEM.

TO find the meaſure of a ſpheric angle.

Around the angular point as a pole deſcribe the projection of a great circle (Prop. XVII.). The portion of this projection intercepted by the ſides of the ſpheric angle, will be the repreſentative of an arch which meaſures the ſpheric angle (6. Cor. Prop. I. B. III.). Wherefore the meaſure of this portion on the circumference of the primitive (Prop. XIX.) will be the meaſure of the ſpheric angle.

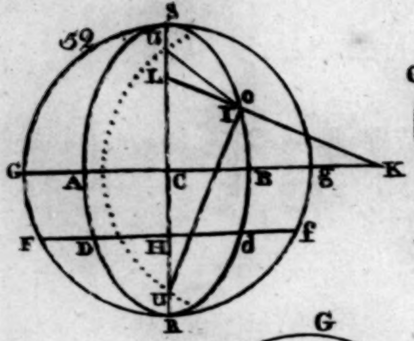
OR, At the angular point draw two right lines touching the projections comprehending the ſpheric angle, the right-lined angles contained by theſe touching lines will be the meaſures of the ſpheric angles comprehended by the projections (Note to Prop. VII.), and therefore of the angles on the ſurface of the ſphere repreſented by the projected angles (PROP. VII.).

BUT if both the projected ſides containing the angle repreſent right circles, the projected angle is a right-lined angle, and meaſures the ſpheric angle repreſented by it (PROP. VII.).

BUT if one of the projected ſides repreſent a right circle, while the other is a circular projection, a right line touching the circular projection will make with the projected right circle right-lined angles, which are the meaſures of the projected ſpheric angles (PROP. VII. and NOTE to the ſame.).

OR, FROM the angular point to two of the projected poles of the ſides draw right lines, the angles comprehended by theſe right lines will be the meaſures of the ſpheric angles (COR. PROP. XIII.).

OR, LASTLY, if both the projected ſides repreſent great circles inclined obliquely to the primitive, and therefore be circular projections; from the centres of the projections to the angular point draw right lines, the angles comprehended by theſe right lines ſhall be the meaſures of the projected ſpheric angles (1. COR. PROP. VII.).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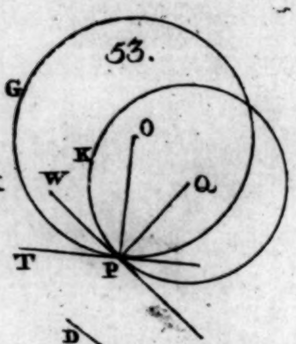
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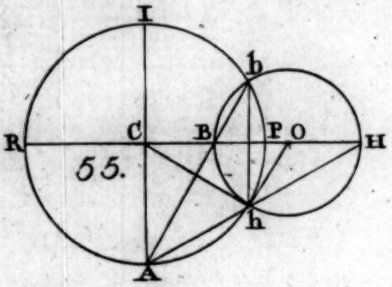
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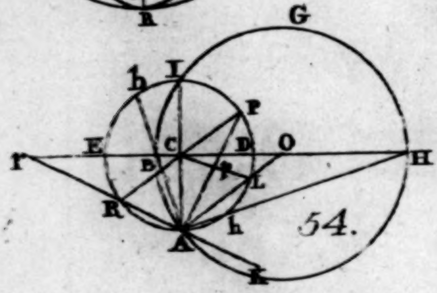
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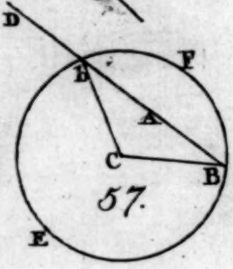
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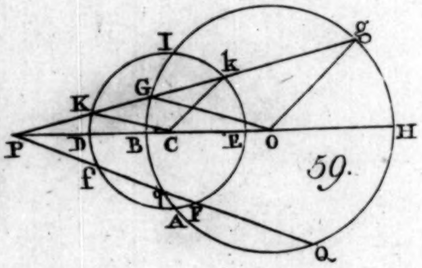
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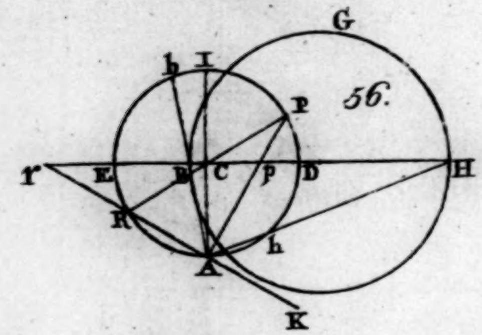
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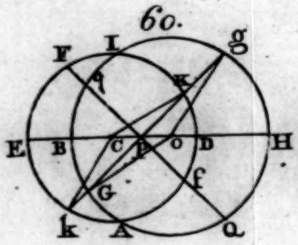
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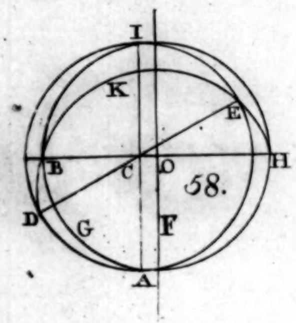
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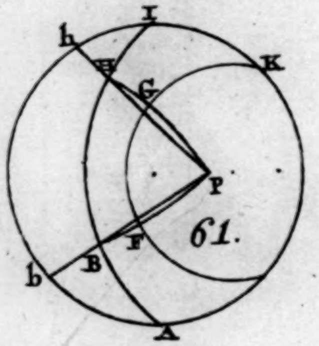
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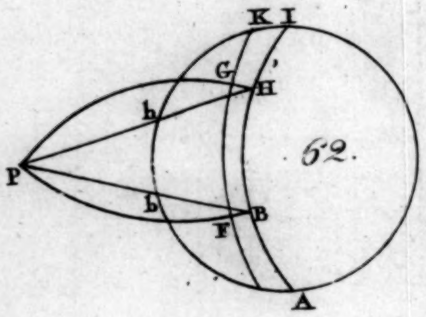
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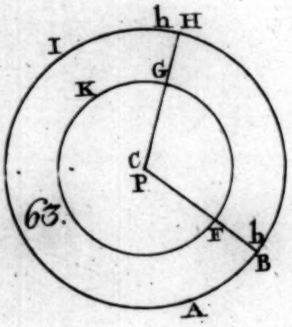
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PROP. XXI. PROBLEM.

THROUGH a given point on the plane of the primitive, to deſcribe the projection of a circle parallel to a circle repreſented by a given projection.

Let I be the given point, AFB the given projection. *Fig.* 66. Find G a pole of AFB (Prop. XVI.), through G, I deſcribe the projection of a great circle (Prop. XVIII.), and finding an arch of the primitive equal to the arch repreſented by GI (Prop. XIX.), round G as a projected pole deſcribe RS the projection of a circle, whoſe diſtance from its pole ſhall be equal to the arch of the primitive meaſuring the diſtance GI (Prop. XVII.). This circle ſhall paſs through the point I, and repreſent a circle parallel to the circle repreſented by AFB.

Because G is the projected pole of the circle of the ſphere repreſented by RS, and GI repreſents the diſtance of this circle from its pole, therefore the point I repreſents a point in the circumference of this circle, and conſequently is in the projection RS. Wherefore RS does paſs through the point I, and becauſe it has the ſame projected pole with the circle AFB, it repreſents a circle parallel to AFB (9. DEF. B. II.).

SCHOL. IF the circle AFB be the projection of a right circle, the primitive will paſs through its poles, and therefore RS will repreſent a circle perpendicular to the primitive (PROP. III. B. II.). Wherefore the centre of the projection RS will be at a diſtance from the centre of the primitive equal to the ſecant of the arch which is the meaſure of GI, and the ſemidiameter of the projection will be equal to the tangent of the ſame meaſure (PROP. IX.).

PROP. XXII. PROBLEM.

THROUGH a given point in the plane of the primitive to deſcribe the projection of a great circle perpendicular to a great circle repreſented by a given projection.

FIND either pole of the given projection (PROP. XVI.), through which and the given point deſcribe a great projection (PROP. XVIII.). Becauſe this projection repreſents a great circle paſſing through the poles of the great circle repreſented by the given projection, the great circles are therefore perpendicular to each other (PROP. III. B. II.).

PROP.

PROP. XXIII. PROBLEM.

THROUGH a given point in the plane of the primitive to deſcribe the projection of a great circle, which ſhall have a given inclination to a great circle repreſented by a given projection. But it is neceſſary that the arch meaſuring the given inclination be not leſs than the complement of the diſtance of the given point from the nearer pole of the given projection, nor greater than this diſtance increaſed by a quadrant.

CASE I. WHEN the given point is the centre of the primitive, and the given projection is that of a right circle.

THROUGH the centre of the primitive draw a right line making with the given rectilinear projection an angle equal to the given inclination; and this right line ſhall repreſent the projection required. For the projection being a diameter of the primitive repreſents a great circle (PROP. III.), and becauſe it makes with the given projection an angle equal to the given inclination, the angle on the ſurface of the ſphere made by the great circles repreſented by them will be equal thereto.

CASE 2. WHEN the given point is in the circumference of the primitive, and the given projection paſſes through the given point.

Fig. 64. IF the given projection be of a right circle, as ACI, and paſs through the given point I in the circumference of the primitive AEID. Draw ECD perpendicular to AI, which will be the line of meaſures to every great circle paſſing through I (PROP. III.). Draw IH making with IA the angle HIA equal to the given inclination, and IG perpendicular to IH meeting ED in G, round the centre G with the diſtance GI deſcribe the circle IF, this ſhall repreſent a great circle, and make with IA an angle equal to the given inclination. Join GA, becauſe IC is equal to CA, and CG common, and the angle ICG equal to the angle GCA, the baſe IG is equal to the baſe GA (4. E. 1.); wherefore the point A is in the circle IF, and conſequently IFA is the projection of a great circle (PROP. III.). But becauſe IH is perpendicular to IG, it touches the circle IFA in the point I (Cor. 16. E. 3.), wherefore the angle FIA is equal to the angle HIA (NOTE to PROP. VII.), viz. to the given inclination.

IF the given projection be not of a right circle, but paſs through the given point I, as ABI, then the ſame things remaining, ED is the line of meaſures of every great projection paſſing through I, and therefore the centres of ABI, and of the projection required are in ED (1. Cor. Prop. II.). To O the centre of ABI draw IO, and IG making therewith the angle GIO equal to the given inclination. Let it meet ED in G, round which with the diſtance GI deſcribe the circle IFA. This repreſents a great circle, for the ſame reaſon as in the preceding, and becauſe to the interſection I of the projections IBA, IFA are drawn the right lines OI, GI, the ſpheric angle repreſented by BIF is meaſured by the right-lined angle OIG (1. Cor. Prop. VII.), viz. by the given inclination.

If the given projection be the primitive itſelf, the Problem is the ſame; for IG is drawn in like manner making with IC the angle GIC equal to the given inclination, and for the ſame reaſon the ſpheric angle repreſented by EIF is meaſured by the rectilineal angle CIG, viz. by the given inclination.

Case 3. When the given point is not in the circumference of the primitive, but is in the circumference of the given projection.

If the given projection be of a right circle, let ECD be the diameter of the primitive repreſenting the right circle, and F the given point therein. Draw RS the locus of the cen- *Fig.* 65.

tres of all the great projections paſſing through the point F (2. COR. PROP. XV.), and the right line FP making with EFD the angle EFP equal to the given inclination. Draw FS perpendicular to FP and round S, the point in which it meets RS, with the diſtance SF deſcribe the circle OFQ, which ſhall be the projection required. For becauſe the centre of the circle OFQ is in the right line RS; OFQ repreſents a great circle (3. COR. PROP. XV.). And becauſe FP is perpendicular to FS, it touches the circle OFQ, therefore the projected ſpheric angle EFO is meaſured by the right-lined angle EFP, viz. by the given inclination.

BUT if BFG be any other given projection of a great circle, find its centre R in the locus RS, and joining RF, draw FS making with RF the angle RFS equal to the given inclination; let it meet RS in S, and round the centre S with the diſtance SF deſcribe the circle OFQ. This circle repreſents a great circle (3. COR. PROP. XV.), and becauſe from the centres R, S are drawn RF, SF to the interſection F, the projected ſpheric angle BFO is meaſured by the rectilineal angle RFS (1. COR. PROP. VII.), viz. by the given inclination.

CASE 4. WHEN the given point is not in the given projection.

LET AB be the given projection, and P the given point. Round P as a projected pole deſcribe the great projection OQ (PROP. XVII.), and finding G the nearer pole of AB (PROP. XVI.), round G as a projected pole deſcribe the projection RS, repreſenting a circle whoſe diſtance from its pole is equal to the arch of the primitive meaſuring the given inclination (PROP. XVII.). Draw the great projection PG (PROP. XVIII.), meeting OQ in K, k. The arches PK, Pk do each repreſent a quadrant, wherefore GK repreſents the complement of the diſtance PG, viz. of the given point from the nearer pole of the given projection, and Gk repreſents the ſame diſtance PG increaſed by the quadrant Pk. The arch meaſuring the given inclination muſt not therefore be leſs than GK, nor greater than Gk. If it be equal to either of them, as to GK; round K as a projected pole deſcribe the great projection DE, which ſhall paſs through P, and F being the point in which it meets AB, the projected ſpheric angle BFE ſhall be meaſured by the given inclination. Becauſe K is the pole of the great projection DE, every point whoſe diſtance from K is meaſured by a quadrant will be in the projection DE (3. COR. PROP. IV. B. II.), but P being the pole of the great projection OKQ, the diſtance KP is meaſured by a quadrant, wherefore the point P is in the projection DE. And becauſe PG paſſes through the poles P, K of OQ, DE, it will be perpendicular to each of them, and therefore each *Fig.* 66. n. 1. 2. 66. n. 1.

of them paſſes through the poles of PG (Prop. III. B. II.), viz. their interſection F is one of the projected poles of PG. Let PG meet AB in L. Becauſe P, G are the poles of OQ, AB, the arches PK, GL repreſent each a quadrant, wherefore PK is equal to GL, and the arch PG being common to each, the whole or remaining arch GK is equal to the whole or remaining arch PL. But the arch GK meaſures the given inclination, and F being the pole of PG, the arch PL meaſures the angle BFE (6. Cor. Prop. VI. B. III.); therefore an arch equal to the meaſure of the given inclination is the meaſure alſo of the projected ſpheric angle BFE.

Fig. 66. n. 2. If the arch meaſuring the given inclination be not equal to GK, or to Gk, it muſt be greater than the one, and leſs than the other, viz. the circle RS is at a diſtance from its pole greater than the diſtance GK, but leſs than the diſtance Gk, and therefore muſt meet OQ. Let it meet it in I, round which point as a projected pole deſcribe the great circle DE (Prop. XVII.). This ſhall be the projection required. Let it meet AB in F, and through G, I deſcribe a great circle meeting AB in L, and DE in H. Then by the ſame reaſoning as in the laſt inſtance is it demonſtrated that DE paſſes through P, that the arches HL, GI are equal, that F is a pole of GI, and therefore that the angle

BFE is meaſured by the arch HL, viz. by an arch equal to the meaſure of the given inclination.

Cor. If the arch meaſuring the given inclination be equal to GK or to Gk, the Problem admits only of one ſolution. If it be greater than GK, but leſs than Gk, it admits of two ſolutions. But if it be leſs than GK, or greater than Gk, the Problem is impoſſible.

Because the great circle repreſented by PG paſſes through the poles, and alſo through the points of concourſe of the circles repreſented by OQ, RS, they will touch each other in the point of concourſe (Cor. Prop. XIII. B. II.), and therefore the projections OQ, RS alſo touch each other in the projected point of concourſe K or k (2. Cor. Prop. VII.). Wherefore the projections OQ, RS meeting each other only in one point, round this point only as a pole can a great projection be deſcribed to anſwer the conditions of the Problem.

But if the arch meaſuring the given inclination, viz. if the meaſure of the diſtance of the circle RS from its pole P, be greater than GK, but leſs than Gk, then the projection RS falls partly without, partly within the projection OQ. The projections therefore cutting each other, meet in two points I, i, round either of which points as a pole a great projection being deſcribed will anſwer the conditions of the Problem.

LASTLY, if the arch meaſuring the given inclination, viz. if the meaſure of the diſtance of the circle RS from its pole P be leſs than GK, or greater than Gk, it will fall wholly within, or wholly without the projection OQ, becauſe GK repreſents the leaſt, and Gk the greateſt diſtance of the point repreſented by G from the circle repreſented by OQ (COR. PROP. V. B. II.). Wherefore the projections OQ, RS not meeting each other at all, the interſections I, i vaniſh, and the Problem is impoſſible.

SCHOLIUM.

BEING a general ſolution of PROP. XVI. B. IV.

THE Stereographic ſolution of this Problem leads alſo to the general ſolution of it in the Orthographic projection.

Fig. 68. LET ARBS be the elliptical projection of a great circle on the plane of the primitive GRgS, and P a point given within the primitive, through which it is required to deſcribe the projection of a great circle, whoſe inclination to the great circle repreſented by RAS may be equal to a given angle, or be meaſured by a given arch.

Let RCS be the major axis of the elliptical projection, and ACB the minor axis.

Because the line of meaſures of every circle of the ſphere is the ſame both in the orthographic and ſtereographic projection, and each projection of a great circle meets the primitive in two points which are in a diameter of the primitive at right angles to the line of meaſures (5. Cor. Prop. V. B. IV. and Prop. III. B. V.), therefore each projection of the ſame great circle paſſes through the ſame points in the circumference of the primitive. ACB therefore is the line of meaſures both in the orthographic and ſtereographic projection of the great circle repreſented by RAS, and each projection paſſes through the ſame points R, S in the circumference of the primitive. And becauſe the orthographic projection of any point of the ſphere, and the point itſelf are in a right line perpendicular to the plane of the primitive, the great circle paſſing through this right line will be perpendicular to the plane of the primitive (18. E. 11.); and therefore will be a right circle (Def. VII. B. IV.). But the ſtereographic projection of the ſame point is in the plane of this right circle (Def. IV. B. IV.), therefore each projection of the ſame point is in the common ſection of this right circle with the primitive, viz. in the ſame diameter of the primitive. Wherefore the point A and the ſtereographic projection of the point repreſented by A are each in the diameter ACB of the primitive. But the point repreſented by A, and for the ſame reaſon every point on the ſurface of the ſphere, is orthographically projected at a diſtance from C the centre of the primitive, equal to the ſine of the point's diſtance from the oppoſite pole of the primitive (1. COR. PROP. V. B. IV.), and the ſame point is ſtereographically projected at a diſtance from the ſame centre equal to the ſemitangent of the ſame diſtance (PROP. IV.). Wherefore draw AE at right angles to AC, meeting the primitive in E, and join RE meeting AC in a. Becauſe AC is the ſine of the arch SE, and aC is the ſemitangent of the ſame arch, the point a will be the ſtereographic place of the point repreſented by A. The ſtereographic projection of the great circle repreſented by RAS paſſes through the point a, and it has been ſhewn that it paſſes through the points R, S; wherefore it is given, viz. the circle deſcribed through R, a, S is this ſtereographic projection. For the ſame reaſon the ſtereographic projection of the point repreſented by P, is given. Let p be this point, through which deſcribe IpL the ſtereographic projection of a great circle, which ſhall have the given inclination to the great circle repreſented by RaS, and let it meet the primitive in I, L. Through I, P, L deſcribe the orthographic projection of a great circle (PROP. XV. B. IV.), this ſhall be the projection required. For the orthographic projections IPL, RAS, and the ſtereopraphic projections IpL, Ras repreſent the ſame great circles, and theſe great circles have the given inclination to each other.

OF

SPHERIC TRIGONOMETRY.

BOOK VI.

DEFINITIONS.

I. A RIGHT line drawn through one extremity of an arch of a circle, perpendicular to the diameter paſſing through the other extremity, is called the SINE of that arch; or of the angle at the centre of the circle ſubtended by the arch, and meaſured thereby.

II. THE ſegment of a diameter paſſing through one extremity of an arch, and intercepted between that extremity and the ſine drawn from the other extremity, is called the VERSED SINE of that arch, or of the angle meaſured thereby.

III. A RIGHT line touching a circle in one extremity of an arch, and intercepted by the diameter drawn through the other extremity, is called the TANGENT of that arch, or of the angle meaſured thereby.

IV. THE ſegment of the diameter drawn through the other extremity, and intercepted between the centre and the tangent, is called the SECANT of the arch, or of the angle meaſured thereby.

V. THE difference between any arch and a quadrant, or between any angle and a right angle, is called the COMPLEMENT of that arch or angle.

VI. THE difference between any arch and a ſemicircle, or between any angle and two right angles, is called the SUPPLEMENT of that arch or angle.

VII. THE ſine, tangent, ſecant of the complement of an arch or angle, are called the CO-SINE, CO-TANGENT, CO-SECANT of that arch or angle.

VIII. THE verſed ſine of the ſupplement of an arch or angle, is called the CO-VERSED SINE of that arch or angle.

COR. 1. THE radius of the circle is the ſine of a quadrant or right angle.

2. THE ſine, tangent and ſecant, coſine, cotangent and coſecant of any arch or angle, are the ſame with the ſine, tangent and ſecant, coſine, cotangent and coſecant of the ſupplement of that arch or angle.

LET AID be an arch of the circle AIK, through the centre C draw the diameter AB, and joining CD, draw DE, CH perpendicular to AB, DF perpendicular to CH, and GDH touching the circle in D. Then DB is the ſupplement of AID, and from the definitions it appears, that DE, DG, CG, DF, DH, CH are equally the ſine, tangent, ſecant, coſine, cotangent, coſecant of the arch AID as of the arch DB. *Fig.* 69. 70.

3. IN an arch leſs than a quadrant, the exceſs of the radius above the verſed ſine, but in an arch greater than a quadrant, the exceſs of the verſed ſine above the radius, is equal to the coſine of the arch.

THE ſame things remaining as in the laſt, in the arch AD leſs than a quadrant AE is the verſed ſine, and CE is the exceſs of the radius CA above AE. But becauſe FE is a parallelogram, CE is equal to DF the coſine of AD. 69.

But in the arch BD greater than a quadrant, the exceſs of its verſed ſine BE above the radius CB is CE, which as before is equal to DF.

4. The difference of the verſed ſines of two arches, which are each leſs or each greater than a quadrant, is the ſame with the difference of the coſines. But if the arches be the one leſs, the other greater than a quadrant, the difference of the verſed ſines is the ſame with the ſum of the coſines.

Thus EL is the difference of the verſed ſines of the arches AD, AK, or BD, BK which are either each leſs or each greater than a quadrant; and it is alſo the difference between CE, CL which are equal to the coſines of the ſame arches.

But of the arches AD, AP the one leſs, the other greater than a quadrant, ER the difference of their verſed ſines is the ſum of CE, CR which are equal to the coſines of AD, AP.

L E M. XII.

Fig. 71. IF four right lines be proportional, viz. AB be to BC as DE to EF, and BG, EI be any other

ther two right lines, the rectangle ABG ſhall be to the rectangle CBG as the rectangle DEI is to the rectangle FEI.

Let AB, BC, be placed in one ſtraight line, and DE, EF in another, and BG be perpendicular to AB, EI to DE; alſo compleat the rectangles AG, CG, DI, FI. Becauſe the rectangles AG, CG have the ſame attitude BG, they are proportional to their baſes (1. E. 6.), viz. the rectangle AG is to the rectangle CG as AB is to BC. For the ſame reaſon the rectangle DI is to the rectangle FI as DE is to FE. But AB is to BC as DE to FE, therefore, ex æquo, the rectangle AG is to the rectangle CG as the rectangle DI is to the rectangle FI.

Cor. 1. If four right lines be proportional, and alſo four others, the rectangles under the antecedents ſhall be proportional to the rectangles under the conſequents.

The ſame things remaining, let BG alſo be to BH as EI to EK; I ſay the rectangle ABG ſhall be to the rectangle CBH, as the rectangle DEI is to the rectangle FEK. Let BG, BH be in one ſtraight line at right angles to AB, and EI, EK in one ſtraight line at right angles to DE, and compleat the rectangles CH, FK. Becauſe AB is to BC as DE to EF, and BG, EI are two other ſtraight lines, the rectangle AG is to the rectangle CG as the rectangle DI is to the rectangle FI. And becauſe BG is to BH as EI is to EK, and BC, EF are two other ſtraight lines, as the rectangle CG is to the rectangle CH ſo is the rectangle FI to the rectangle FK; wherefore ex æquo, the rectangle AG is to the rectangle CH as the rectangle DI is to the rectangle FK.

COR. 2. IF four right lines be proportional, viz. A be to B as C to D, and alſo E be to F as G to C, then the rectangle under A, E is to the rectangle under B, F as G is to D.

FOR the rectangle under A, E is to that under B, F as the rectangle under C, G is to that under C, D. But the rectangle under C, G is to that under C, D as G is to D (1. E. 6.), therefore, ex æquo, the rectangle under A, E is to that under B, F as G is to D.

LEM. XIII.

HALF the ſupplement of an arch is the ſame with the complement of half the arch.

LET AGD be an arch of a circle, draw the diameter AB, and BD will be the ſupplement of AD (6. DEF.). Biſect the arches AD, BD in F, E; and I ſay DE is the complement of FD. Becauſe the arch AGD is double to the arch FD, and the arch BD is double to the arch DE, the whole or remaining arch AGB is double to the whole or remaining arch FE. But the arch AGB is a ſemicircle or double to a quadrant, wherefore the arch FE is a quadrant, and therefore DE which is the difference between the arch FD, and the quadrant FE will be the complement of FD (5. DEF.). *Fig.* 72. 73.

COR. HENCE the ſine and coſine of half the ſupplement of an arch is the coſine and ſine of half the arch.

LEM. XIV.

THE difference between one arch and the ſupplement of another is equal to the ſupplement of the ſum of the two arches.

LET AG, GD be two arches of a circle, draw the diameter AB, then GB is the ſupplement of AG, and DB is the ſupplement of the whole arch AGD. But the arch DB is the difference between the arches GD, GB, therefore the ſupplement of the ſum of the arches AG, GD is the 72. 73.

the ſame with the difference between one of the arches GD, and the ſupplement GB of the other.

Cor. Hence the ſine and coſine of half the difference between any arch and the ſupplement of another are the coſine and ſine of half the ſum of the arches.

This is only the application of the corollary of the preceding.

LEM. XV.

THE difference between two arches is the ſame with the ſupplement of the ſum of either of them and the ſupplement of the other.

Fig. 72. 74. Let AD, DG be two arches of a circle, whoſe difference is AG. Draw the diameter AB, then DB is the ſupplement of AD, and GB is the ſum of GD, DB. But AG is the ſupplement of the whole arch GDB, therefore the difference of the arches is the ſame with the ſupplement of the ſum of either of them GD, and the ſupplement of the other AD.

Cor. Hence the ſine and coſine of half the difference between two arches are the coſine and ſine of half the ſum of either of the arches and the ſupplement of the other.

This alſo is a direct application of the Cor. Lem. XIII.

LEM. XVI.

The ſecant of any arch is to the tangent of the ſame, as the radius is to the ſine of the arch.

Let AID be any arch of a circle, whoſe centre is C, draw the diameter AB, and DE perpendicular to it, alſo draw DG touching the circle in D, and meeting AB in G. Then GC is the ſecant, GD the tangent, and DE the ſine of the arch AID. But becauſe CD is perpendicular to DG, the triangles CGD, CDE are equiangular (8. E. 6.), wherefore GC is to GD as CD is to DE. Fig. 69. 70.

Cor. The rectangle under the ſecants of two arches is to the rectangle under the tangents of the ſame, as the ſquare of radius is to the rectangle under the ſines of the arches.

The ſame things remaining, let CS, BS, PR be the ſecant, tangent and ſine of any other arch BP of the ſame circle. Becauſe CG is to GD as CD to DE, and CS is 69.

to BS as CP or CD to PR, therefore the rectangle under CG, CS is to that under GD, BS as the ſquare of CD is to the rectangle under DE, PR (1. COR. LEM. XII.).

LEM. XVII.

THE coſine of any arch is to the ſine of the ſame, as the radius is to the tangent of the arch.

Fig. 69. 70. THE ſame things remaining as in the preceding, CE is equal to the coſine of the arch AID, and the triangles CDE, CGD being equiangular, CE is to ED as CD is to DG.

COR. 1. THE ſine of an arch is to the coſine as radius is to the cotangent.

THIS is the propoſition itſelf aſſerted of the complement.

COR. 2. THE rectangle under the ſine and cotangent of any arch is to the rectangle under the ſine and cotangent of any other arch as the coſine of the firſt arch is to the coſine of the ſecond.

69. FOR, the ſame things remaining as in the corollary of

the preceding, becauſe DE is to CE as CD is to DH, and PR is to CR as CD is to PT the cotangent of the arch BP, therefore the rectangle under DE, DH is equal to the rectangle under CD, CE, and the rectangle under PR, PT is equal to the rectangle under CD, CR (16. E. 6.). Wherefore the rectangle under DE, DH is to the rectangle under PR, PT, (as the rectangle under CD, CE is to that under CD, CR, viz.) as CE is to CR (1. E. 6.).

LEM. XVIII.

THE radius is a mean proportional between the tangent and cotangent, between the ſine and coſecant, and between the coſine and ſecant of any arch.

The ſame things remaining, draw CH perpendicular to AB meeting GD in H, and draw DF perpendicular to CH; then DF is the coſine, CH the coſecant, DH the cotangent of the arch AID, and CF is equal to DE the ſine of the ſame. But becauſe the angle GCH is right, and CD is perpendicular to GH, the radius CD is a mean proportional between the tangent GD and the cotangent DH (Cor. 8. E. 6.). *Fig.* 69. 70.

Also, becauſe GDC is a right angle, and DE is perpendicular to GC, the radius CD is a mean proportional between the ſecant GC and the coſine CE. And becauſe HDC is a right angle, and DF perpendicular to HC, the radius CD is a mean proportional between the coſecant HC and the ſine CF (Cor. 8. E. 6.).

Cor. 1. Hence the tangents of any two arches are inverſely as their cotangents.

Let A, B repreſent any two arches of a circle, becauſe the radius is a mean proportional between the tangent and cotangent of A, and alſo between the tangent and cotangent of B, therefore ex æquo perturbate, the tangent of A is to the tangent of B as the cotangent of B is to the cotangent of A.

Cor. 2. For the ſame reaſon, the ſines of any two arches of a circle are inverſely as the coſecants, and the coſines inverſely as the ſecants.

LEM. XIX.

THE ſum of the ſines of two arches is to their difference, as the tangent of half the ſum of the arches is to the tangent of half their difference.

Let AD, AB be two arches of the circle ADB, draw the diameter AC, and DH, BG perpendicular to AC; produce DH to meet the circumference in F, and draw FL, DP perpendicular to BG. Becauſe DF is perpendicular to AC, it is biſected in H (3. E. 3.), wherefore LP is alſo biſected in G, and LG, GP are each equal to FH or HD; but BG, DH are the ſines of the arches AB, AD, therefore BL is equal to the ſum, and BP to the difference of the ſines BG, DH. Join BD, and draw COE perpendicular to BD, alſo joining CD, draw EIK to touch the circle in E, and meet CD, CA in I, K. The arches BD, DF are biſected in E, A (30. E. 3.), wherefore becauſe the arch BD is double to ED, and the arch DF double to DA, the whole arch BF is double to the whole arch EA. But the arch AF being equal to the arch AD, the whole BF is the ſum of the arches AB, AD, and BD is their difference, therefore AE is equal to half the ſum, and DE to half the difference of the arches AB, AD. But EK, EI are the tangents of the arches AE, DE; and I ſay BL is to BP as EK is to EI. *Fig.* 75.

Produce BD to meet AC, FL in M, N. Becauſe PL is biſected in G, DN will be biſected in M (Lem. IV.), and

BD is biſected in O (3. E. 3.); wherefore BD being double to DO, and DN double to DM, the whole BN is double to the whole OM. Therefore BN is to BD as OM is to OD. But becauſe of the parallels, BN is to BD as BL to BP, and OM is to OD as EK is to EI (LEM. V.). Therefore, ex æquo, BL is to BP as EK is to EI.

COR. IF the difference between two arches be biſected, the leſs arch increaſed by half the difference, or the greater diminiſhed by half the difference, is equal to half the ſum of the arches.

FOR it is demonſtrated that AE is half the arch BF, viz. half the ſum of the arches EF, EB, or EF, ED. And it is alſo equal to the leſs arch ED increaſed by AD or to the greater arch leſſened by the arch AF; while AD, AF are each equal to half the arch DF, which is the difference of the arches EF, ED.

LEM. XX.

THE ſum of the tangents of two arches is to their difference, as the ſine of the ſum of the arches is to the ſine of their difference.

Let AE, ED be two arches of the circle ADB, and every thing remaining as in the preceding, let CB be drawn, meeting IE in R. Becauſe BD is biſected in O, IR will be biſected in E (Lem. V.), wherefore RK is equal to the ſum of the tangents EK, EI, and IK is their difference. Alſo becauſe the arch BD is biſected in E, the arch AB is equal to the ſum, and AD is the difference of the arches AE, ED, I ſay therefore, that RK is to IK as the ſine BG is to the ſine DH. *Fig.* 75. 76.

For becauſe of the parallels, RK is to IK (as BM is to DM, viz.) as BG is to DH.

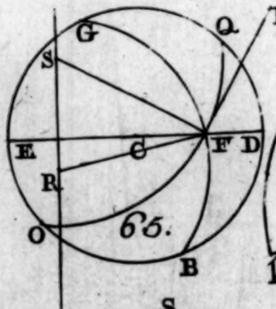
L E M. XXI.

If two angles be of the ſame affection the ſum of their ſecants is to their difference; but if the angles be of different affection, the difference of their ſecants is to their ſum, as the cotangent of half their ſum is to the tangent of half their difference.

Let ICD, LCD be two angles at the centre C of the circle IDL, of the ſame affection (*Fig.* 77.), but of different affection (*Fig.* 78.). Draw ADB touching the circle in 77. 78.

in D, and meeting CI, CL in A, B; then CA, CB are the ſecants of the angles ICD, LCD. Round the centre C with the diſtance CB deſcribe a circle meeting AC, in F, G, and AB in E. Becauſe CF, CG are each equal to CB, therefore (in *Fig.* 77.) AG is equal to the ſum, and AF to the difference of the ſecants AC, CB; but the reverſe in *Fig.* 78. Join CE, BF, BG, and draw FH parallel to AB. Becauſe CD meets AB at right angles, it will biſect it (3. E. 3.), wherefore BD is equal to DE, alſo CD is common to the triangles BCD, ECD, and the angles BDC, EDC are equal, being each a right one; therefore the correſpondent angles BCD, ECD are equal between themſelves (4. E. 1.), and conſequently the angle ICE is equal to the difference between the angles ICD, LCD. But the angle ICE at the centre is double to the angle (EBF at the circumference, viz. to the alternate angle) BFH; wherefore the angle BFH is equal to half the difference of the angles ICD, LCD. For the ſame reaſon the angle BGF is half the angle ICL, viz. half the ſum of the angles, and becauſe the angle FBG in a ſemicircle is a right one, the angle BFG is the complement of BGF. Therefore the angle BFG is the complement of half the ſum of the angles ICD, LCD. I ſay then that AG is to AF as the tangent of the angle BFG is to the tangent of the angle BFH.

Because FB is perpendicular to BG, the right lines BG, BH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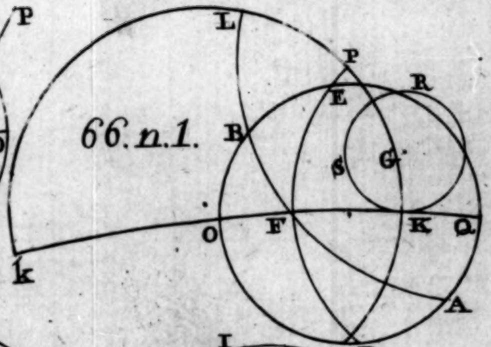
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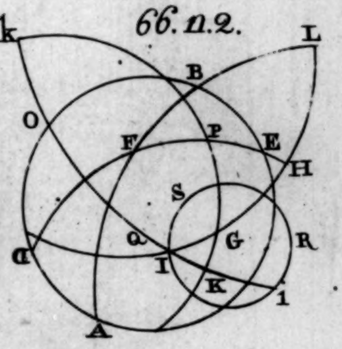
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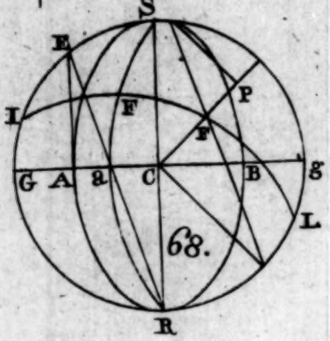
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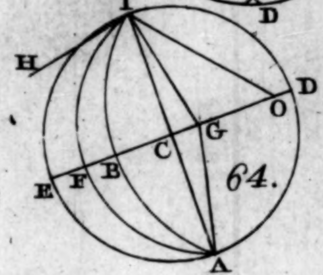
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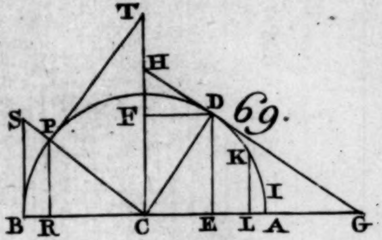
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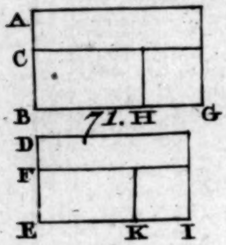
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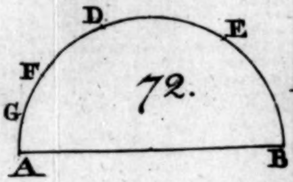
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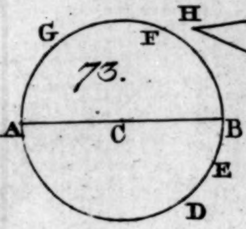
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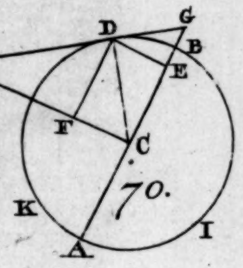
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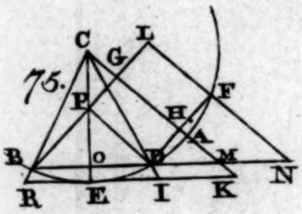
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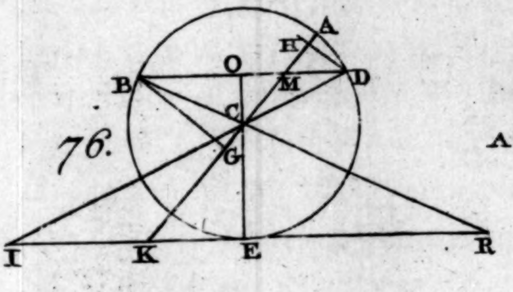
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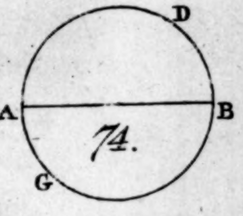

74.

TAB. VII. *pag.* 250.

BH will be proportional to the tangents of the angles BFG, BFH. But becauſe FH is parallel to AB; BG, BH will be proportional alſo to AG, AF; wherefore, ex æquo, AG, AF are proportional to the tangents of the angles BFG, BFH.

COR. IF from one of the angles of a plane triangle, a perpendicular be drawn to the oppoſite ſide, the ſum of the ſides is to the difference as the cotangent of half the ſum of the angles at the vertex (made by the perpendicular) is to the tangent of half their difference.

IN the plane triangle ACB, CD perpendicular to the baſe AB falls within the triangle, and the angles ACD, BCD being the ſame with the angles ICD, LCD, it has been demonſtrated that the ſum of AC, CB is to their difference as the cotangent of half the ſum of the angles ACD, BCD is to the tangent of half their difference. *Fig.* 77.

BUT if the perpendicular fall without the triangle, join FE, then the quadrilateral BEFG being inſcribed in a circle, the angle EFG is equal to the external angle ABH (COR. 22. E. 3.), viz. to the angle BHF. But the angle EFG is half the angle ACE, viz. half the ſum of the angles ACD, BCD. Therefore the angle BHF is equal to half the ſum of theſe angles, and the angle BFG is half the 78.

angle ACB, or the difference of the angles ACD, BCD. It has then been demonſtrated that the ſum of AC, CB is to their difference as the cotangent of the angle BHF is to the tangent of the angle BFG.

LEM. XXII.

THE difference of the verſed ſines of any two arches is to the verſed ſine of any other arch, as the rectangle under the ſines of half the ſum and half the difference of the arches is to the ſquare of the ſine of half the other arch.

Fig. 79. 80.

LET AB, AD be two arches, and AE any other arch of the circle ABa; draw the diameter ACa, and BF, DG, EH perpendicular to Aa; alſo join AE, BD, and draw CPS, COI perpendicular to them, meeting the circumference in S, I, and draw IK at right angles to Aa. Then AF, AG are the verſed ſines of the arches AB, AD, and FG is the difference of the verſed ſines. And becauſe CI drawn through the centre is perpendicular to BD, the arch BD, and the right line BD are biſected in I, O (30. and 3. E. 3.). For the ſame reaſon the arch AE, and the right line AE, are biſected in S, P; therefore BI is half the difference of the arches AB, AD, and BO is its ſine; alſo AS is half the arch AE, and AP is its ſine, and AH is the verſed ſine of AE. Laſtly, becauſe BD the difference of the arches AB, AD is biſected in I, AI is half the ſum of AB, AD, (Cor. Lem. XIX.), and IK is its ſine. I ſay then, that FG is to AH as the rectangle under IK, BO is to the ſquare of AP.

Because the angles DOC, DGC are right, the four points D, O, G, C are in a circle; wherefore the angle ODG is equal to the angle OC[illegible], A and conſequently the triangles BDL, ICK are equiangular. Therefore IC is to IK as BD is to BL or FG, and the rectangle under IC, FG (is equal to the rectangle under IK, BD, viz.) is double to the rectangle under IK, BO. But becauſe Aa is double to IC, the rectangle under Aa, FG is (double to the rectangle under IC, FG, and therefore) quadruple to the rectangle under IK, BO. Alſo, the rectangle aAH (is equal to the ſquare of AE (Cor. 8. E. 6.), viz. becauſe AE is biſected in P,) is quadruple to the ſquare of AP. Wherefore the rectangle under Aa, FG is to the rectangle under Aa, AH, as the rectangle under IK, BO is to the ſquare of AP. But the rectangle under Aa, FG is to the rectangle under Aa, AH as FG is to AH (1. E. 6.), therefore FG is to AH as the rectangle under IK, BO is to the ſquare of AP.

Cor. 1. The difference of the verſed ſines of two arches

AB, AD is to the coverſed ſine of a third arch aE as the rectangle under the ſines of half the ſum and half the difference of the arches AB, AD is to the ſquare of the coſine of half the arch aE.

For the arch AE being the ſupplement of aE, the coverſed ſine of aE is AH the verſed ſine of AE; and the coſine of half the arch aE is the ſine of half the arch AE (Cor. Lem. XIII.). But by this Prop. the difference of the verſed ſines of AB, AD is to the verſed ſine of AE, viz. to the coverſed ſine of aE, as the rectangle under the ſines of half the ſum and half the difference of the arches AB, AD is to the ſquare of the ſine half the arch AE, or ſquare of the coſine of half the arch aE.

Cor. 2. The difference between the verſed ſine of one arch and the coverſed ſine of another is to the verſed ſine of a third arch, as the rectangle under the coſines of half the ſum and half the difference of the arches is to the ſquare of the ſine of half the third arch.

Let AB, aD be the two arches, then the difference between the verſed ſine of AB and the coverſed ſine of aD is the difference between the verſed ſines of AB, AD. And the coſines of half the ſum and half the difference of the arches AB, aD are the ſines of half the difference and half the ſum of the arches AB, AD (Cor. Lem. 14. and 15.). Therefore this corollary is the propoſition itſelf in other words.

Cor. 3. The difference between the verſed ſine of one arch and the coverſed ſine of a third arch is to the coverſed ſine of a third arch, as the rectangle under the coſines of half the ſum and half the difference of the two arches is to the ſquare of the coſine of half the third arch.

This is included, and explained in Cor. 1. and 2.

LEM. XXIII.

THE rectangle under the ſine of any arch and the tangent of half the arch is double to the ſquare of the ſine of half the arch.

Let AB be any arch of the circle ABa, draw the diameter ACa, and BD, AF at right angles to it. Join AB, perpendicular to which draw CH, meeting the circumference in E, and AF in F. Becauſe the arch AB is biſected in E (30. E. 3.) AF is the tangent, and AH the ſine of half the arch AB, alſo BD is the ſine of the whole AB. I ſay therefore *Fig.* 81.

fore that the rectangle under BD, AF is double to the ſquare of AH.

DRAW BG perpendicular to AF, becauſe the angles BGF, BHF are each right, the four points B, G, F, H are in a circle (CONVERSE. 22. E. 3.); wherefore the rectangle GAF is equal to the rectangle BAH (COR. 36. E. 3.). But the oppoſite ſides AG, BD of the rectangle GD are equal between themſelves, therefore the rectangle under AG, AF is equal to the rectangle under BD, AF, and becauſe AB is double to AH, the rectangle BAH is double to the ſquare of AH. Wherefore the rectangle under BD, AF is double to the ſquare of AH.

COR. HENCE if A, B denote any two arches of a circle, the rectangle under the ſine of A and the tangent of half A, is to the rectangle under the ſine of B, and the tangent of half B in the duplicate ratio of the ſines of half the arches A, B.

FOR the rectangles are reſpectively double to the ſquares of the ſines, and therefore are proportional to the ſquares of the ſines.

L E M. XXIV.

IN a plane quadrilateral figure if the exceſs of the ſquares of two of the adjacent ſides be together equal to the exceſs of the ſquares of the remaining two adjacent ſides, the diagonals ſhall meet each other at right angles.

Fig. 82.

LET ABCD be a trapezium, AC, BD its diagonals meeting each other in E, and let the exceſs of the ſquares of AB, AD be equal to the exceſs of the ſquares of BC, CD; I ſay, AC is perpendicular to BD. If not, draw AF perpendicular to BD, and join CF. Becauſe the exceſs of the ſquares of AB, AD is equal to the exceſs of the ſquares of BC, CD, but the exceſs of the ſquares of AB, AD is equal to the exceſs of the ſquare of BD above twice the rectangle BDF (13. E. 2.); therefore the exceſs of the ſquares of BC, CD is equal to the exceſs of the ſquare of BD above twice the rectangle BDF. Wherefore CF is perpendicular to BD (CONVERSE. 13. E. 2.), and conſequently AFC is one ſtraight line (14. E. 1.). But AEC is alſo a ſtraight line, wherefore two ſtraight lines encloſe a ſpace, which is abſurd. AF therefore is not perpendicular to BD, nor for the ſame reaſon, any other right line drawn from A, unleſs AEC.

AND in the ſame manner, if the perpendicular AF were ſuppoſed to fall without the triangle ABD, would the propoſition be demonſtrated from the 12. E. 2.

PROP. I. THEOREM.

IF there be a ſpheric triangle, and a plane quadrilateral figure be formed, two of whoſe ſides are the ſecants, the other two the tangents of two of the ſides of the ſpheric triangle, and the angle comprehended by the ſecants be meaſured by the ſpheric baſe; the angle comprehended by the tangents ſhall be the meaſure of the ſpheric angle oppoſite to the baſe, the diagonals of the quadrilateral ſhall interſect each other at right angles, the ſegments of the diagonal joining the angle of the ſecants and the angle of the tangents ſhall be the ſecant and tangent of the ſpheric perpendicular drawn from the vertical angle to the baſe, the angles which this diagonal makes with the ſecants ſhall be meaſured by the ſpheric ſegments of the baſe, and the angles which this diagonal makes with the tangents ſhall be the meaſures of the ſpheric angles which the perpendicular makes with the ſides.

LET KPG be a ſpheric triangle on the ſurface of the ſphere KPCG, whoſe centre is C. Draw the right lines CK, CG, which will be in the plane of the baſe KG, and PW, PT touching the circles PK, PG in P, and meeting CK, CG in W, T. Then CW, CT are the ſecants, and PW, PT the tangents of the arches PK, PG; alſo the angle WCT is meaſured by the arch KG (3. DEF. B. III.). Compleat the quadrilateral WCTZ, whoſe remaining ſides WZ, TZ are equal to WP, TP; I ſay firſt, that the angle WZT is the meaſure of the ſpheric angle KPG. *Fig.* 24.

JOIN WT, becauſe the ſides WZ, TZ are equal to the ſides WP, TP, and the baſe WT is common to the triangles WZT, WPT, the angle WZT ſhall be equal to the angle WPT, and the angle PWT to the angle ZWT (8. E. 1.). But the angle WPT is the meaſure of the ſpheric angle KPG (AXIOM. B. III.), therefore the ſpheric angle KPG is alſo meaſured by the angle WZT.

IF the other diagonal CZ be joined, I ſay it ſhall meet WT at right angles. Becauſe the angle CPT is right, the exceſs of the ſquare of CT above the ſquare of PT is equal to the ſquare of CP. But PT is equal to ZT, therefore the exceſs of the ſquare of CT above the ſquare of ZT is equal to the ſquare of CP. For the ſame reaſon the exceſs of the ſquare of CW above the ſquare of WZ is equal to the ſame ſquare of CP; therefore the exceſs of the ſquares of CT, TZ is equal to the exceſs of the ſquares of CW, WZ, and conſequently the diagonals CZ, WT of the quadrilateral CWZT meet each other at right angles (LEM. XXIV.).

AGAIN, let the plane PCZ cut the ſphere in the great circle EPD, meeting the baſe KG in D, and V being the concourſe of the diagonals; I ſay the arch PD is perpendicular to the baſe KG, and that the ſegments CV, VZ are the ſecant and tangent of the arch PD. Join PV, which will be the common ſection of the planes PCZ, WPT. Becauſe WP is equal to WZ, and WV is common, and the correſpondent angles PWV, ZWV of the triangles TPW, TZW have been ſhewn to be equal, the remaining ſides PV, VZ, and the angles WVP, WVZ, as alſo WPV, WZV, are equal between themſelves (4. E. 1.). But the angle WVZ is a right angle, therefore the angle WVP is alſo a right angle, and conſequently the right line WV being perpendicular to the right lines CV, PV, is perpendicular to the plane CPV (4. E. 11.), and CV being the common ſection of the planes CPV, CKG, the planes are themſelves perpendicular to each other (4. DEF. E. 11.). Wherefore the arch PD is perpendicular to the arch KG (1. COR. PROP. I. B. III.). But becauſe the right lines WP, PT touch the cirles KP, PG, the plane WPT touches the ſphere in P (1. Cor. Prop. VI. B. V.); wherefore PV touches the circle EPD in P (Prop. X. B. II.), and conſequently CV, PV are the ſecant and tangent of the arch PD. But PV has been ſhewn to be equal to VZ, therefore CV, VZ are equal to the ſecant and tangent of the ſpheric perpendicular PD.

Lastly, the angles WCZ, TCZ are meaſured by the ſpheric ſegments KD, GD. And becauſe the angles WPV, TPV are the meaſures of the ſpheric angles KPD, GPD, but the angle WZC has been ſhewn to be equal to the angle WPV, as for the ſame reaſon is the angle TZC equal to the angle TPV, therefore the angles WZC, TZC meaſure the ſpheric angles KPD, GPD.

Cor. 1. If the ſpheric triangle KPG be projected upon the plane of the baſe KG, viz. into the triangle KpG, the quadrilateral WCTZ will be ſituated in that plane, the angle C will be the centre of the primitive, and PD being a great circle perpendicular to the primitive will be projected into the diameter CpD, the common ſection of the right circle PD with the primitive (Prop. I. B. V. and Cor.), and therefore be in the ſame right line with the diagonal CZ of the quadrilateral.

Cor. 2. If there be a right-angled ſpheric triangle, and a plane triangle be conſtituted, whoſe ſides are the ſecant and tangent of the ſpheric hypothenuſe, and the angle at the baſe ſubtended by the tangent be meaſured by one of the ſpheric ſides about the right angle; the other angle at the baſe will meaſure the ſpheric angle ſubtended by the ſame ſide, and if a perpendicular be drawn to the baſe of the plane triangle, the ſegments of the baſe will be equal to the ſecant and tangent of the remaining ſide of the right-angled ſpheric triangle.

Fig. 83. Thus in the plane triangle WCZ, the ſides WC, WZ are the ſecant and tangent of PK the hypothenuſe of the right-angled ſpheric triangle KPD, the angle WCZ is meaſured by the ſpheric ſide KD; while the angle WZC is the meaſure of the ſpheric angle KPD oppoſite to the ſide KD, and the ſegments CV, VZ made by the perpendicular WV are the ſecant and tangent of the other ſpheric ſide PD.

And what is demonſtrated of the ſpheric ſide KD and the oppoſite angle KPD, would in the ſame terms be demonſtrated of the ſide PD and the oppoſite angle PKD, if a plane touching the ſphere in K projected a plane triangle on the plane of the circle PD.

Schol. This propoſition includes in itſelf the ſolution of every problem in ſpheric trigonometry, and from this alone, with the aid of plane geometry are all its theorems readily deduced.

PROP. II. THEOREM.

THE ſines of the ſides of a ſpheric triangle are proportional to the ſines of the oppoſite angles.

Case 1. When the triangle is right-angled.

Let KPD be a ſpheric triangle, right-angled at D; I ſay the ſines of the ſides are proportional to the ſines of the oppoſite angles. *Fig.* 83.

Let the triangle be projected upon the plane of the ſide KD, and every thing remain as in the 2. Cor. Prop. I. Then in the plane triangle WCZ, WZ is to WC as the ſine of the angle WCZ is to the ſine of the angle WZC. That is, the tangent of the hypothenuſe PK is to the ſecant of the ſame as the ſine of the ſide KD is to the ſine of the angle KPD (2. Cor. Prop. I.). But the tangent of an arch is to the ſecant of the ſame as the ſine of the arch is to radius (Lem. XVI.); therefore the ſine of the hypothenuſe PK is to radius or the ſine of the right angle PDK (1. Cor. Def.), as the ſine of KD is to the ſine of the angle KPD.

For the ſame reaſon, if the triangle be projected upon the plane of the ſide PD, and a plane triangle conſtituted as in the preceding inſtance, will the ſine of PK be to radius as the ſine of PD is to the ſine of the angle PKD.

Wherefore, ex æquo, the ſine of KD is to the ſine of the angle KPD, as the ſine of PD is to the ſine of the angle PKD.

Case 2. Let KPG be any ſpheric triangle; I ſay the ſine of any one ſide KP is to the ſine of the oppoſite angle PGK, as the ſine of any other ſide GP is to the ſine of the oppoſite angle PKG.

Draw the arch PD perpendicular to KG. Then in the right-angled triangles PDK, PDG, the ſine of KP is to radius as the ſine of PD is to the ſine of the angle PKG, and the ſine of GP is to radius as the ſine of PD is to the ſine of the angle PGK. Wherefore, ex æquo perturbate, the ſine of KP is to the ſine of the angle PGK as the ſine of GP is to the ſine of the angle PKG.

Cor. If through one of the angles of a ſpheric triangle a great circle be drawn making equal angles with the ſides, and meet the baſe; the ſines of the ſides ſhall be proportional to the ſines of the ſegments of the baſe.

85. 86. Let ABC be a ſpheric triangle, and through an angle A

let a great circle AE be drawn to make equal angles (internally or externally) with the ſides AB, AC, and meet the baſe BC in D; I ſay the ſine of AB is to the ſine of AC as the ſine of BD is to the ſine of CD.

When the equal angles are external, the angle CAD is equal to the angle BAE, therefore adding the common angle BAC, the whole angle BAD is equal (to the whole angle CAE, viz.) to the ſupplement of the angle CAD. And when AE falls within the triangle, the angle ADB is the ſupplement of the angle ADC. Therefore the angles BAD, ADB, are either the ſame with the angles CAD, ADC, or equal to them, or the ſupplements of them, and conſeqently have the ſame or equal ſines (2. Cor. Def.). But the ſine of AB is to the ſine of BD as the ſine of the angle ADB is to the ſine of the angle BAD, and the ſine of AC is to the ſine of CD (as the ſine of the angle ADC is to the ſine of the angle CAD, viz.) as the ſine of the angle ADB is to the ſine of the angle BAD. Wherefore, ex æquo, the ſine of AB is to the ſine of BD as the ſine of AC is to the ſine of CD.

DEFINITIONS.

OF THE CIRCULAR PARTS.

IX. EVERY triangle is made up of ſix parts, viz. three ſides and three angles, and as in right-angled triangles, one of theſe ſix parts is determined, there remain only five which are the ſubject of enquiry. Of theſe five, the two ſides about the right angle, with the complements of the hypothenuſe and the two acute angles are called the FIVE CIRCULAR PARTS.

X. IF any one of theſe five be eſteemed a MIDDLE PART, the two adjoining on each ſide of this middle part are denominated EXTREMES CONJUNCT OR ADJACENT; and the remaining two EXTREMES DISJUNCT OR OPPOSITE.

PROP. III. THEOREM.

IN any right-angled ſpheric triangle, the ſine of middle part and radius are reciprocally proportional to the tangents of the extremes conjunct, and the coſines of the extremes disjunct.

NOTE. As three of the circular parts are complements, therefore when this propoſition demands the ſine, coſine or tangent of any of theſe three, the coſine, ſine and tangent of the arch or angle of which each of theſe circular parts is the complement, is underſtood thereby.

All the variety of this noble propoſition is comprehended in three caſes; when a ſide, the complement of the hypothenuſe, or the complement of an acute angle is the middle part. The demonſtration of this propoſition will therefore conſiſt in the demonſtration of each of theſe three caſes.

Case I. Let KPD be a ſpheric triangle, having a right angle at D, and let a ſide PD be a middle part, then the ſide KD, and the complement of the angle KPD are extremes conjunct, the complements of the angle PKD and of the hypothenuſe PK are extremes disjunct. I ſay therefore that radius is to the tangent of KD as the cotangent of the angle KPD is to the ſine of PD; and that radius is to the ſine of PK as the ſine of the angle PKD is to the ſine of PD. *Fig.* 83.

The ſame things remaining as in the 2. Cor. Prop. I., in the plane triangle WCZ, becauſe WV is perpendicular to CZ, therefore CV, VZ are the tangents of the angles CWV, ZWV, or the cotangents of the angles WCZ, WZC to the radius WV, and conſequently are proportional to the cotangents of theſe angles to any other radius. Wherefore the cotangent of the angle WCZ is to the cotangent of the angle WZC, as CV is to VZ. But the angles WCZ, WZC are the meaſures of the ſpheric ſide KD and the angle KPD, alſo CV, VZ are the ſecant and tangent of the ſpheric ſide

PD (2. Cor. Prop. I.), wherefore the cotangent of KD is to the cotangent of the angle KPD (as the ſecant of PD is to the tangent of PD, viz.) as radius is to the ſine of PD (Lem. XVI.). But the cotangent of KD is to radius as radius is to the tangent of KD (Lem. XVIII.), therefore, ex æquo, and by alternation, radius is to the tangent of KD as the cotangent of the angle KPD is to the ſine of PD. And, radius is to the ſine of PK as the ſine of the angle PKD is to the ſine of PD (Prop. II.).

Case 2. The complement of the hypothenuſe PK being middle part, the extremes conjunct are the complements of the angles PKD, KPD, and the extremes disjunct are the ſides KD, PD. I ſay therefore, that radius is to the cotangent of the angle PKD as the cotangent of the angle KPD is to the coſine of PK; and radius is to the coſine of PD as the coſine of KD is to the coſine of PK.

Through C the pole of KD deſcribe the great circle CE perpendicular to KP (Prop. XXII. B. V.), and let it meet KD in F. Becauſe the great circle CE paſſes through the pole of KD, it is perpendicular to KD (Prop. III. B. II.), wherefore the great circles KP, KD being each perpendicular to the ſame great circle CE, they do each paſs through the poles of CE, and conſequently their interſection K is one of the poles. Wherefore the arch FE meaſures the angle PKD (4. Cor. Prop. VII. B. V.), as for the ſame reaſon the angle PCE is meaſured by the arch DF; and becauſe CF, CD, KE, KF are quadrants, CE is the complement of FE, viz. of the meaſure of the angle PKD, KD is the complement of DF, viz. of the meaſure of the angle PCE, and PE, PC are the complements of the arches KP, PD. But in the right-angled ſpheric triangle CEP, the ſide PE being middle part, radius is to the tangent of CE as the cotangent of the angle CPE is to the ſine of PE (Case 1. of this Prop.), that is, radius is to the cotangent of the angle PKD as the cotangent of the angle KPD is to the coſine of PK.

And, in the ſame triangle, radius is to the ſine of PC as the ſine of the angle PCE is to the ſine of PE (Prop. II.), viz. radius is to the coſine of PD as the coſine of KD is to the coſine of PK.

Case 3. The complement of an acute angle, viz. of the angle PKD being middle part, the ſide KD and the complement of the hypothenuſe PK are extremes conjunct, while the ſide PD and the complement of the angle KPD are the extremes disjunct. I ſay therefore, that radius is to the cotangent of PK as the tangent of KD is to the coſine of the angle PKD; and radius is to the ſine of the angle KPD as the coſine of PD is to the coſine of the angle PKD.

The ſame things remaining as in Case 2., the ſide EC of the right-angled ſpheric triangle CEP being middle part, radius is to the tangent of EP as the cotangent of the angle PCE is to the ſine of EC (Case 1.), that is, radius is to the cotangent of PK as the tangent of KD is to the coſine of the angle PKD.

And, in the ſame triangle, radius is to the ſine of the angle CPE as the ſine of CP is to the ſine of EC (Prop. II.), viz. radius is to the ſine of the angle KPD as the coſine of PD is to the coſine of the angle PKD.

PROP. IV. THEOREM.

If from any angle of a ſpheric triangle a perpendicular be drawn to the oppoſite ſide, the tangents of the ſegments of the baſe are as the tangents of the angles at the vertex made by the perpendicular.

Fig. 83. 84. Let KPG be a ſpheric triangle, and from the angle P be drawn PD perpendicular to the oppoſite ſide KG; the tangent of KD is to the tangent of GD as the tangent of the angle KPD is to the tangent of the angle GPD.

Every thing remaining the ſame as in the 1. Cor.

Prop. I. in the plane triangles WCT, WZT, WV, VT are the tangents of the angles WCZ, TCZ to the radius CV, and of the angles WZC, TZC to the radius ZV, and therefore are proportional to the tangents of theſe angles to any other radius. Wherefore the tangent of the angle WCZ is to the tangent of the angle TCZ (as WV is to TV, viz.) as the tangent of the angle WZC is to the tangent of the angle TZC. That is, the tangent of KD is to the tangent of GD as the tangent of the angle KPD is to the tangent of the angle GPD (Prop. I.).

PROP. V. THEOREM.

THE ſame things remaining, the coſines of the ſides are as the coſines of the ſegments of the baſe.

I say, the coſine of KP is to the coſine of GP as the coſine of KD is to the coſine of GD. *Fig.* 83. 84.

In the plane triangle WCT, TC is to WC (as the ſine of the angle CWT is to the ſine of the angle WTC, viz. becauſe the angles CWT, WTC are the complements of the angles WCZ, TCZ), as the coſine of the angle WCZ is to the coſine of the angle TCZ. That is, the ſecant of GP is to the ſecant of KP as the coſine of KD is to the coſine

ſine of GD (PROP. I.). But the ſecants of two arches are inverſely as their coſines (2. COR. LEM. XVIII.), therefore, ex æquo, the coſine of KP is to the coſine of GP as the coſine of KD is to the coſine of GD.

PROP. VI. THEOREM.

THE cotangents of the ſides are as the coſines of the angles at the vertex made by the perpendicular.

Fig. 83. 84. I SAY, the cotangent of KP is to the cotangent of GP as the coſine of the angle KPD is to the coſine of the angle GPD.

IN the plane triangle WZT, TZ is to WZ as the coſine of the angle WZC is to the coſine of the angle TZC. That is, the tangent of GP is to the tangent of KP as the coſine of the angle KPD is to the coſine of the angle GPD (PROP. I.). But the tangents of two arches are inverſely as their cotangents (1. COR. LEM. XVIII.), therefore, ex æquo, the cotangent of KP is to the cotangent of GP as the coſine of the angle KPD is to the coſine of the angle GPD.

PROP.

PROP. VII. THEOREM.

THE ſame things remaining, if the ſides be of the ſame affection, the cotangent of half their ſum is to the tangent of half their difference; but if of different affection, the tangent of half their difference is to the cotangent of half their ſum, as the cotangent of half the ſum of the ſegments of the baſe made by the perpendicular is to the tangent of half their difference.

I say, the cotangent of half the ſum of KP, GP is to the tangent of half the difference between KP, GP, or the tangent of half the difference between KP, GP is to the cotangent of half the ſum of KP, GP (accordingly, as KP, GP are of the ſame or different affection), as the cotangent of half the ſum of the ſegments KD, GD is to the tangent of half their difference. *Fig.* 83. 84.

In the plane triangle WCT, the ſum of the ſides WC, CT is to their difference as the cotangent of half the ſum of the angles WCV, TCV is to the tangent of half their difference (Cor. Lem. XXI.); that is, the ſum of the ſecants of KP, GP is to their difference as the cotangent of half the ſum of the ſegments KD, GD is to the tangent of half their difference. But the ſum of the ſecants of two arches is to their difference as the cotangent of half the ſum of the arches is to the tangent of half their difference, or as the tangent of half the difference of the arches is to the cotangent of half their ſum, accordingly as the arches are of the ſame or different affection (LEM. XXI.); therefore the cotangent of half the ſum of KP, GP is to the tangent of half their difference, or the tangent of half the difference between KP, GP is to the cotangent of half their ſum (accordingly as the ſides KP, GP are of the ſame, or different affection), as the cotangent of half the ſum of the ſegments KD, GD is to the tangent of half their difference.

COR. 1. IF the perpendicular fall within the triangle, and the ſides be of the ſame affection, as the cotangent of half their ſum is to the tangent of half their difference, ſo is the cotangent of half the baſe to the tangent of half the difference of the ſegments of the baſe.

FOR the baſe is then the ſum of the ſegments.

COR. 2. BUT if the perpendicular fall without, while the ſides are of the ſame affection, the cotangent of half their ſum is to the tangent of half their difference as the cotangent of half the baſe is to the tangent of half the ſum of the ſegments of the baſe.

For the baſe is then the difference of the ſegments, but the ſides being of the ſame affection, the cotangent of half their ſum is to the tangent of half their difference (as the cotangent of half the ſum of the ſegments is to the tangent of half the baſe, viz.) as the cotangent of half the baſe is to the tangent of half the ſum of the ſegments (1. Cor. Lem. XVIII.).

Cor. 3. If the perpendicular fall within the triangle, but the ſides are of different affection, the cotangent of half their ſum is to the tangent of half their difference, as the tangent of half the baſe is to the cotangent of half the difference of the ſegments of the baſe.

For the baſe being then the ſum of the ſegments, by inverſion of the propoſition, the cotangent of half the ſum of the ſides is to the tangent of half their difference (as the tangent of half the difference of the ſegments is to the cotangent of half the baſe, viz.) as the tangent of half the baſe is to the cotangent of half the difference of the ſegments (1. Cor. Lem. XVIII.).

Cor. 4. If the perpendicular fall without, while the ſides are of different affection, the cotangent of half their ſum is to the tangent of half their difference as the tangent of half the baſe is to the cotangent of half the ſum of the ſegments of the baſe.

FOR the baſe being here the difference of the ſegments, this is merely the inverſion of the propoſition.

PROP. VIII. THEOREM.

THE ſine of the ſum of the ſides of any ſpheric triangle, is to the ſine of their difference as the cotangent of half the ſum of the angles at the vertex, made by the perpendicular, is to the tangent of half their difference.

THE ſame things remaining, I ſay, the ſine of the ſum of the ſides KP, GP is to the ſine of their difference, as the cotangent of half the ſum of the angles KPD, GPD is to the tangent of half their difference.

Fig. 83. 84. IN the plane triangle WZT, the ſum of the ſides WZ, ZT is to their difference, as the cotangent of half the ſum of the angles WZV, TZV is to the tangent of half their difference (COR. LEM. XXI.); that is, the ſum of the tangents of KP, GP is to their difference, as the cotangent of half the ſum of the angles KPD, GPD is to the tangent of half their difference (PROP. I.). But the ſum of the tangents of two arches is to their difference, as the ſine of their ſum is to the ſine of their difference (LEM. XX.); therefore the ſine of the ſum of KP, GP is to the ſine of their difference, as the cotangent of half the ſum of the angles KPD, GPD is to the tangent of half their difference.

COR. THE ſine of the ſum of the ſides of a ſpheric triangle is to the ſine of their difference, as the cotangent of half the vertical angle is to the tangent of half the difference or of half the ſum of the angles at the vertex, made by the perpendicular, accordingly as the perpendicular falls within or without the triangle.

IF the perpendicular fall within, the baſe being then the ſum of the ſegments, this is the propoſition itſelf.

BUT if the perpendicular fall without, the baſe is the difference of the ſegments. Wherefore the ſine of the ſum of the ſides is to the ſine of their difference, (as the cotangent of half the ſum of the ſegments is to the tangent of half the baſe, viz.), as the cotangent of half the baſe is to the tangent of half the ſum of the ſegments of the baſe (1. COR. LEM. XVIII.).

SCHOL. WHEN in any future propoſition it is left in doubt whether the difference or ſum of the angles at the vertex, the difference or ſum of the ſegments of the baſe (made by a perpendicular drawn thereto from the oppoſite angle) be underſtood, the meaning is, that the difference is to be uſed, if the perpendicular fall within the triangle; but the ſum, if it fall without.

PROP. IX. THEOREM.

IN any ſpheric triangle, as the tangent of half the ſum of two of the ſides is to the tangent of half the difference, ſo is the tangent of half the ſum of the angles at the baſe to the tangent of half their difference.

Fig. 83. 84. IN the ſpheric triangle KPG, the tangent of half the ſum of any two ſides KP, PG is to the tangent of half their difference, as the tangent of half the ſum of the angles PKG, PGK is to the tangent of half their difference.

THE ſine of KP is to the ſine of GP as the ſine of the angle PGK is to the ſine of the angle PKG (PROP. II.), therefore (componendo and dividendo) the ſum of the ſines of KP, GP is to their difference as the ſum of the ſines of the angles PGK, PKG is to their difference. But the ſum of the ſines of two arches is to their difference, as the tangent of half the ſum of the arches is to the tangent of half their difference (LEM. XIX.); wherefore the tangent of half the ſum of KP, GP is to the tangent of half their difference, as the tangent of half the ſum of the angles PGK, PKG is to the tangent of half their difference.

Cor. In a ſpheric triangle, if a perpendicular be drawn from one of the angles upon the oppoſite ſide, the ſquare of the ſine of half the ſum of the ſides is to the ſquare of the ſine of half their difference, as the rectangle under the cotangent of half the vertical angle of the triangle and the tangent of half the ſum of the angles at the baſe is to the rectangle under the tangent of half the difference or half the ſum of the angles at the vertex and the tangent of half the difference of the angles at the baſe.

For the ſine of the ſum of the ſides is to the ſine of their difference as the cotangent of half the vertical angle is to the tangent of half the difference or half the ſum of the vertical angles, made by the perpendicular (Cor. Prop. VIII.). And the tangent of half the ſum of the ſides is to the tangent of half their difference as the tangent of half the ſum of the angles at the baſe is to the tangent of half their difference. Therefore the rectangle under the ſine of the ſum and the tangent of half the ſum of the ſides is to the rectangle under the ſine of their difference and the tangent of half the difference of the ſides, as the rectangle under the cotangent of half the vertical angle and the tangent of half the ſum of the angles at the baſe is to the rectangle under the tangent of half the difference or half the ſum of the angles at the vertex and the tangent of half the difference of the angles at the baſe (1. COR. LEM. XII.). But the rectangle under the ſine of one arch and the tangent of half the ſame is to the rectangle under the ſine of another arch and the tangent of half the ſame, as the ſquare of the ſine of half the firſt arch is to the ſquare of the ſine of half the ſecond arch (COR. LEM. XXIII.). Wherefore the corollary is immediately inferred by equality of proportion.

PROP. X. THEOREM.

IN any ſpheric triangle, if a perpendicular be drawn from one of the angles to the oppoſite ſide, the coſines of the angles at the baſe are proportional to the ſines of the angles at the vertex made by the perpendicular.

Fig. 32. IN the ſpheric triangle ABC, if from one of the angles at A be drawn AD perpendicular to BC, the coſine of the angle ABC is to the coſine of the angle ACB as the ſine of the angle BAD is to the ſine of the angle CAD.

LET *a*bc be the ſemi-ſupplemental triangle, and from the correſpondent angle *a* be drawn *a*d perpendicular to bc, the coſine of *a*c is to the coſine of *a*b, as the coſine of cd is to the coſine of bd (Prop. V.). But *a*c, *a*b are the meaſures of the angles ABC, ACB (Prop. III. B. III.), and cd, bd are the complements of the meaſures of the angles BAD, CAD (Cor. Prop. II. B. III.), wherefore the coſine of the angle ABC is to the coſine of the angle ACB, as the ſine of the angle BAD is to the ſine of the angle CAD.

PROP. XI. THEOREM.

IN a ſpheric triangle, the cotangents of the angles at the baſe are proportional to the ſines of the ſegments of the baſe, made by a perpendicular drawn thereto from the oppoſite angle.

The ſame things remaining as in the preceding, I ſay, the cotangent of the angle ABC is to the cotangent of the angle ACB, as the ſine of the ſegment BD is to the ſine of the ſegment CD. *Fig.* 32.

In the ſemi-ſupplemental triangle *a*bc, the cotangent of *a*c is to the cotangent of *a*b, as the coſine of the angle c*a*d is to the coſine of the angle b*a*d (Prop. VI.). But the ſides *a*c, *a*b, meaſure the angles ABC, ACB, and the complements of the ſegments ~~AD~~, BD, CD, meaſure the angles c*a*d, b*a*d (COR. PROP. II. B. III.); wherefore, the cotangent of the angle ABC is to the cotangent of the angle ACB, as the ſine of AD is to the ſine of BD.

L E M. XXV.

THE cotangent of half the ſum of the complements of two arches is to the tangent of half the difference of the complements as the tangent of half the ſum of the arches is to the tangent of half their difference.

Fig. 87. 88. LET AD, AE be two arches of the circle ADB, draw the diameters AB, FG at right angles to each other, DH parallel to FG, and draw the diameter EI. Becauſe DH is perpendicular to AB, the arch AD is equal to the arch AH; and the arches EF, GI are equal, becauſe ſubtending equal angles at the centre. Wherefore DF or HG is the complement of the arch AD, and EF or GI is the complement of the arch AE. Alſo EH is the ſum of the arches, and DE is their difference.

87. IF then the arches be of the ſame affection, HI, which

is the ſum of the complements, is the ſupplement of HE, and DE the difference of the complements, as alſo of the arches themſelves. Wherefore the cotangent of half the arch HI is the ſame with the tangent of half the arch HE (LEM. XIII.), and conſequently the cotangent of half the arch HI is to the tangent of half the arch DE, as the tangent of half the arch HE is to the tangent of half the arch DE.

BUT if the arches be of different affection, HI, which is the difference of the complements, is the ſupplement of HE, and DE the ſum of the complements is the ſame with the difference of the arches themſelves. Wherefore, the cotangent of half the arch DE is to the tangent of half the arch HI, (as the cotangent of half the arch DE is to the cotangent of half the arch HE, viz.) as the tangent of half the arch HE is to the tangent of half the arch DE (1. COR. LEM. XVIII.). *Fig.* 88.

PROP. XII. THEOREM.

IN a ſpheric triangle, if the angles at the baſe be of the ſame affection, the cotangent of half the ſum is to the tangent of half their difference; but if of different affection, the tangent of half their difference is to the cotangent of half their ſum, as the tangent of half the ſum of the angles at the vertex (made by a perpendicular drawn therefrom to the baſe) is to the tangent of half their difference.

Fig. 32. THE ſame things remaining, I ſay the cotangent of half the ſum of the angles ABC, ACB is to the tangent of half their difference (if the angles be of the ſame affection); but the tangent of half their difference is to the cotangent of half their ſum (if they be of different affection), as the tangent of half the ſum of the angles BAD, CAD is to the tangent of half their difference.

IN the ſemi-ſupplemental triangle *a*bc, the cotangent of half the ſum of *a*c, *a*b is to the tangent of half the difference of *a*c, *a*b; or the tangent of half their difference is to the cotangent of half their ſum (accordingly as the ſides *a*c, *a*b are of the ſame, or different affection), as the cotangent of half the ſum of the ſegments cd, bd is to the tangent of half their difference (PROP. VII.). But the ſides *a*c, *a*b meaſure the angles ABC, ACB (PROP. III. B. III.), and therefore are of the ſame affection with them. Alſo cd, bd are the complements of the meaſures of the angles BAD, CAD (COR. PROP. II. B. III.). Wherefore the cotangent of half the ſum of the angles ABC, ACB is to the tangent of half their difference, or the tangent of half their difference is to the cotangent of half their ſum (accordingly as the angles are of the ſame or different affection), as the tangent of half the ſum of the angles BAD, CAD is to the tangent of half their difference (Lem. XXV.).

Cor. In a ſpheric triangle, the tangents of half the ſum and half the dfference of the angles at the baſe are inverſely as the cotangent of half the vertical angle and the tangent of half the difference or half the ſum of the angles at the vertex made by the perpendicular, accordingly as the perpendicular falls within or without the triangle.

If the angles at the baſe be of the ſame affection, the perpendicular falls within the triangle, therefore the vertical angle of the triangle is the ſum of the angles made by the perpendicular, and conſequently by inverſion and alternation of the propoſition, the tangent of half the difference of the angles at the baſe is to the tangent of half the difference of the angles at the vertex, (as the cotangent of half the ſum of the angles at the baſe is to the tangent of half the vertical angle of the triangle, viz.) as the cotangent of half the vertical angle is to the tangent of half the ſum of the angles at the baſe (1. Cor. Lem. XVIII.)

But if the angles be of different affection, the perpendicular

cular falls without the triangle, wherefore the vertical angle of the triangle is the difference of the angles made by the perpendicular, and conſequently by alternation of the propoſition, the tangent of half the difference of the angles at the baſe is to the tangent of half the ſum of the angles at the vertex, (as the cotangent of half the ſum of the angles at the baſe is to the tangent of half the vertical angle of the triangle, viz.) as the cotangent of half the vertical angle is to the tangent of half the ſum of the angles at the baſe (1. Cor. Lem. XVIII.).

PROP. XIII. THEOREM.

IN a ſpheric triangle, the ſine of the ſum of the angles at the baſe is to the ſine of their difference, as the tangent of half the ſum of the ſegments of the baſe is to the tangent of half their difference.

Fig. 32. The ſame things remaining, the ſine of the ſum of the angles ABC, ACB is to the ſine of their difference, as the tangent of half the ſum of the ſegments BD, CD is to the tangent of half their difference.

In the ſemi-ſupplemental triangle *a*bc, the ſine of the ſum of the ſides *a*c, *a*b is to the ſine of their difference, as the cotangent of half the ſum of the angles c*a*d, b*a*d is to the tangent of half their difference (Prop. VIII.). That is, for the ſame reaſon as in the preceding, the ſine of the ſum of the angles ABC, ACB is to the ſine of their difference, as the tangent of half the ſum of BD, CD is to the tangent of half their difference.

Cor. 1. If the angles be of the ſame affection, the ſine of their ſum is to the ſine of their difference, as the tangent of half the baſe is to the tangent of half the difference of the ſegments of the baſe.

Because the angles at the baſe are of the ſame affection, the perpendicular falls within the triangle (Prop. XXII. B. III.), and conſequently the baſe is the ſum of the ſegments. Wherefore this corollary is the direct propoſition itſelf.

Cor. 2 If the angles be of different affection, the ſine of their ſum is to the ſine of their difference, as the cotangent of half the baſe is to the cotangent of half the ſum of the ſegments of the baſe.

For the angles being of different affection, the perpendicular falls without the triangle (Prop. XXII. B. III.), and conſequently the baſe is the difference of the ſegments. Wherefore the ſine of the ſum of the angles at the baſe is to the ſine of their difference, (as the tangent of half the ſum of the ſegments is to the tangent of half the baſe, viz.) as the cotangent of half the baſe is to the cotangent of half the ſum of the ſegments of the baſe (1. Cor. Lem. XVIII.).

PROP. XIV. THEOREM.

THE ſine of half the ſum of two ſides of a ſpheric triangle is to the ſine of half their difference, as the cotangent of half the included angle is to the tangent of half the difference of the angles at the baſe.

Fig. 41. 42. Let ABC be a ſpheric triangle; I ſay, the ſine of half the ſum of AB, AC is to the ſine of half their difference, as the cotangent of half the vertical angle BAC is to the tangent of half the difference of the angles B, C.

Draw AD perpendicular to the baſe BC. The cotangent of half the angle BAC is to the tangent of half the difference of the angles B, C, as the tangent of half the ſum of the angles B, C is to the tangent of half the difference or half the ſum of the angles BAD, CAD (COR. PROP. XII.). Therefore the ſquare of the cotangent of half the angle BAC is to the ſquare of the tangent of half the difference of the angles B, C, (as the rectangle under the cotangent of half the angle BAC and the tangent of half the ſum of the angles B, C is to the rectangle under the tangent of half the difference of the angles B, C and the tangent of half the difference or half the ſum of the angles BAD, CAD (LEM. XII.), viz.) as the ſquare of the ſine of half the ſum of the ſides AB, AC is to the ſquare of the ſine of half their difference (COR. PROP. IX.). Wherefore the ſquares being proportional, the right lines themſelves, of which they are the ſquares, are proportional, viz. the cotangent of half the angle BAC is to the tangent of half the difference of the angles B, C, as the ſine of half the ſum of the ſides AB, AC is to the ſine of half their difference. Which is the propoſition itſelf inverted.

PROP XV. THEOREM.

THE coſine of half the ſum of two ſides of a ſpheric triangle is to the coſine of half their difference, as the cotangent of half the included angle is to the tangent of half the ſum of the angles at the baſe.

THE ſine of half the ſum of the ſides is to the ſine of half their difference, as the cotangent of half the included angle is to the tangent of half the difference of the angles at the baſe (PROP. XIV.). And the tangent of half the difference of the ſides is to the tangent of half their ſum, or what is the ſame, (1. COR. LEM. XVIII.) the cotangent of half the ſum of the ſides is to the cotangent of half their difference, as the tangent of half the difference of the angles at the baſe is to the tangent of half their ſum (PROP. IX.). Therefore the rectangle under the ſine and cotangent of half the ſum of the ſides is to the rectangle under the ſine and cotangent of half the difference of the ſides, as the cotangent of half the included angle is to the tangent of half the ſum of the angles at the baſe (2. COR. LEM. XII.). But the rectangle under the ſine and cotangent of one arch is to the rectangle under the ſine and cotangent of the other arch, as the coſine of the firſt arch is to the coſine of the ſecond arch (2. COR. LEM. XVII.). Wherefore the coſine of half the ſum of the ſides is to the coſine of half their difference, as the cotangent of half the included angle is to the tangent of half the ſum of the angles at the baſe.

PROP. XVI. THEOREM.

IN a ſpheric triangle, the ſine of half the ſum of the angles at the baſe is to the ſine of half their difference, as the tangent of half the baſe is to the tangent of half the difference of the ſides.

Let ABC be a ſpheric triangle, as the ſine of half the ſum of two of the angles B, C is to the ſine of half their difference, ſo is the tangent of half the ſide BC to the tangent of half the difference of the ſides AB, AC. *Fig.* 32.

In the ſemi-ſupplemental triangle *a*bc, the ſine of half the ſum of *a*c, *a*b is to the ſine of half their difference, as the cotangent of half the angle b*a*c is to the tangent of half the difference of the angles c, b (Prop. XIV.). But *a*c, *a*b are the meaſures of the angles B, C, the angles c, b are meaſured by the ſides AB, AC, and the ſupplement of the angle b*a*c is meaſured by the ſide BC (Prop. III. B. III.). Wherefore the cotangent of half the angle b*a*c is the ſame with the tangent of half the ſide BC (Lem. XIII.); and conſequently the ſine of half the ſum of the angles B, C is to the ſine of half their difference, as the tangent of half the ſide BC is to the tangent of half the difference of the ſides AB, AC.

PROP. XVII. THEOREM.

In a ſpheric triangle, the coſine of half the ſum of the angles at the baſe is to the coſine of half their difference, as the tangent of half the baſe is to the tangent of half the ſum of the ſides.

This is inferred from Prop. XV. in the ſame manner as the preceding was deduced from Prop. XIV.

PROP. XVIII. THEOREM.

In a ſpheric triangle, as the rectangle under the ſines of any two ſides is to the ſquare of radius, ſo is the difference between the verſed ſine of the baſe, and the verſed ſine of the difference of the ſides, to the verſed ſine of the angle included by the ſides.

Fig. 89. 90. 91. 92. Let KPG be a ſpheric triangle; I ſay, the rectangle under the ſines of two of its ſides KP, PG, is to the ſquare of radius, as the exceſs of the verſed ſines of KG and the difference of the ſides KP, PG is to the verſed ſine of the angle KPG. On the circumference of the primitive KG, viz. on the circumference of a great circle of the ſphere, make the arch KA equal to the meaſure of the arch KP, and the arch AI, towards the parts of K, equal to the meaſure of the arch GP. Let C be the centre of the ſphere, or of the circle KG; join CK, CI, and draw IE, GL perpendicular to CK. KI is the meaſure of the difference of the ſides KP, PG, and KE is its verſed ſine; alſo KL is the verſed ſine of the baſe KG, and EL is the difference of the verſed ſines KE, KL.

DRAW a right line touching the circle KG in A, and meeting CK, CI in W, B. Becauſe CW is the ſecant (of the arch AK, viz.) of the arch KP, to the radius of the ſphere, let the quadrilateral CWZT be conſtituted as in 1. COR. PROP. I. Then AW, AB are equal to WZ, TZ, viz. to the tangents of the ſides KP, PG, and CB is equal to CT the ſecant of the ſide PG. Draw TO, BF perpendicular to CW, TH to WZ, alſo in the right line AW make AQ equal to ZH, towards the ſame parts of A, that H is of Z, with reſpect to the common point W. Becauſe AB is equal to TZ, BQ will be equal to the ſum or difference of TZ, ZH, accordingly as the angle WZT is greater or leſs than a right angle. But if a circle be deſcribed round the centre Z with the diſtance ZT, ZH will be the coſine, and the ſum or difference of ZT, ZH will be the verſed ſine of the angle WZT, accordingly as the angle is greater or leſs than a right angle (3. COR. DEF.). Wherefore BQ is equal to the verſed ſine of the angle WZT, to the radius ZT or AB, and conſequently is proportional to the verſed ſine of the ſame angle to any other radius; viz. the radius

CG of the ſphere is to the verſed ſine of the angle WZT, (as AB is to BQ, viz.) as the rectangle WAB is to the rectangle under AW, BQ (1. E. 6.). But becauſe TH, ZV are perpendicular to WZ, WT, the four points H, T, V, Z are in a circle. For the ſame reaſon, becauſe TO, CV are perpendicular to WC, WT, the four points O, T, V, C are in a circle; and CA, BF being perpendicular to AW, CW, the four points A, C, B, F are in a circle. Wherefore the rectangle HWZ is equal (to the rectangle TWV, viz.) to the rectangle CWO, and the rectangle AWB is equal to the rectangle CWF (35. or Cor. 36. E. 3.). But becauſe AW is equal to WZ, and AQ to ZH, the whole or remainder WQ is equal to the whole or remainder HW, and the rectangle AWQ equal (to the rectangle ZWH, viz.) to the rectangle CWO. And it has been ſhewn that the rectangle AWB is equal to the rectangle CWF, therefore the whole or remaining rectangle under AW, BQ is equal to the whole or remaining rectangle under CW, FO. But becauſe of the parallels, CE is to CF, (as CI is to CB, viz. as CG is to CT, viz.) as CL is to CO. Wherefore (dividendo or componendo) EL is to FO, (as CL is to CO, viz.) as CG is to CT; and EL is to CG, as the rectangle under CW, FO or AW, BQ is to the rectangle WCT. But it has been ſhewn that CG is to the verſed ſine of the angle WZT, as the rectangle WAB is to the rectangle under AW, BQ. Therefore, ex æquo perturbato, EL is to the verſed ſine of the angle WZT, as the rectangle WAB is to the rectangle WCT. But the angle WZT meaſures the ſpheric angle KPG, WA, AB are equal to the tangents, and WC, CT are the ſecants of the ſpheric ſides KP, PG, and the rectangle under the tangents is to the rectangle under the ſecants of two arches, as the rectangle under the ſines is to the ſquare of radius (Cor. Lem. XVI.). Wherefore, ex æquo, the rectangle under the ſines of KP, PG, is to the ſquare of radius, as EL the difference of the verſed ſine of KG and the verſed ſine of the difference of the ſides KP, PG is to the verſed ſine of the angle KPG.

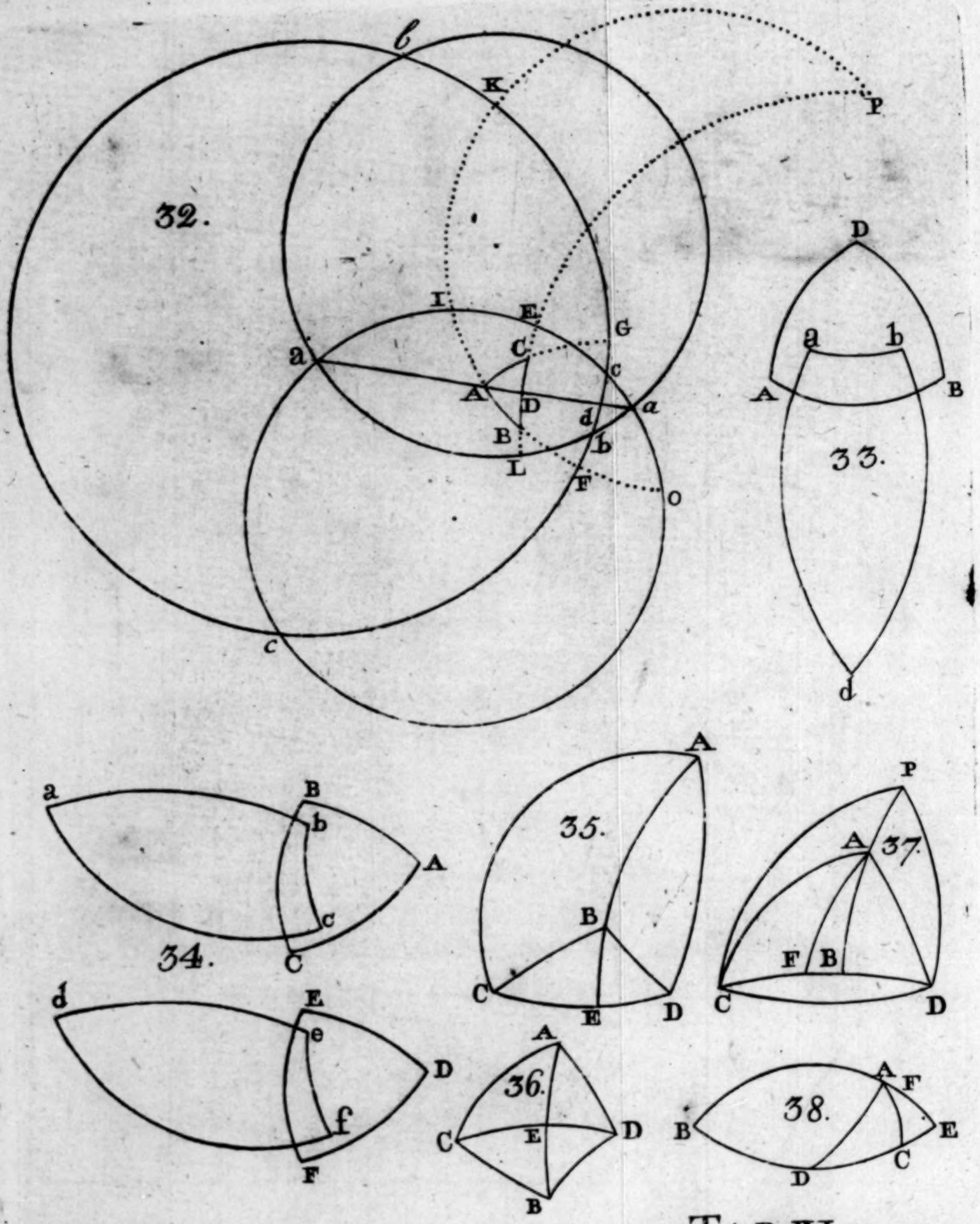
Cor. In a ſpheric triangle, the rectangle under the ſines of two of its angles is to the ſquare of radius, as the difference between the coverſed ſine of the third angle and the verſed ſine of the difference of the two angles is to the coverſed ſine of the ſide included between the two.

I say, the rectangle under the ſines of two of the angles B, C of the ſpheric triangle ABC is to the ſquare of radius, as the difference between the coverſed ſine of the angle A and the verſed ſine of the difference of the angles B, C is to the coverſed ſine of BC. *Fig.* 32.

In the ſemi-ſupplemental triangle *a*bc, the rectangle under the ſines of *a*c, *a*b is to the ſquare of radius, as the difference between the verſed ſine of bc and the verſed ſine of the difference of the ſides *a*c, *a*b is to the verſed ſine of the angle b*a*c. But *a*c, *a*b meaſure the angles B, C, while bc is the meaſure of the ſupplement of the angle A, and the angle b*ac* is meaſured by the ſupplement of the ſide BC (Prop. II. B. III.); conſequently *a*c, *a*b have the ſame ſines with the angles B, C, the verſed ſine of the difference between *a*c, *a*b is the ſame with that of the difference between the angles B, C; but the verſed ſines of bc and the angle b*a*c are the ſame with the coverſed ſines of the angle A and the ſide BC. Therefore the rectangle under the ſines of the angles B, C is to the ſquare of radius, as the difference between the coverſed ſine of the angle A and the verſed ſine of the difference of the angles B, C is to the coverſed ſine of the ſide BC.

PROP. XIX. THEOREM.

IN a ſpheric triangle, the rectangle under the ſines of any two of its ſides is to the ſquare of radius, as the difference between the verſed ſine of the baſe and the verſed ſine of the ſum of the ſides is to the coverſed ſine of the angle included by the ſides.



TAB. IV. *pag.* 297.

Let KP, PG be any two ſides of the ſpheric triangle KPG; I ſay the rectangle under the ſines of KP, PG is to the ſquare of radius, as the difference between the verſed ſine of KG, and the verſed ſine of the ſum of the ſides KP, PG is to the coverſed ſine of the angle KPG. *Fig.* 93. 94. 95. 96.

Let every thing remain the ſame as in the preceding, except that the arch AI is towards the parts of A oppoſite to K, and Q towards the parts of A, with reſpect to W, oppoſite to thoſe that H is of Z. Then KI is the meaſure of the ſum of the ſides KP, PG, whoſe verſed ſine is KE, and EL as before is the difference between the verſed ſines KE, KL. And BQ will be equal to the difference or ſum of ZT, ZH, accordingly as the angle WZT is greater or leſs than a right angle, viz. as the ſupplement of WZT is leſs or greater; and conſequently BQ is equal to the verſed ſine of the ſupplement, or the coverſed ſine of the angle WZT to the radius ZT. With this difference the demonſtration proceeds in the very ſame words.

Cor. 1. In a ſpheric triangle ABC, the rectangle under the ſines of two of its angles B, C is to the ſquare of radius, as the difference between the coverſed ſine of the third angle A and the verſed ſine of the ſum of the angles B, C is to the verſed ſine of the baſe BC. 32.

This is inferred in the ſame manner with the corollary of the preceding from the ſemi-ſupplemental triangle, obſerving that the ſum of the ſides *a*c, *a*b meaſure the ſum of the angles B, C, and therefore have the ſame verſed ſine; and that the coverſed ſine of the angle b*a*c is the ſame with the verſed ſine of BC.

Cor. 2. In a ſpheric triangle, the difference between the verſed ſine of the baſe and the verſed ſine of the difference of the ſides is to the verſed ſine of the vertical angle, as the difference between the verſed ſine of the baſe and the verſed ſine of the ſum of the ſides is to the coverſed ſine of the vertical angle.

This is inferred by direct equality of proportion from this Prop. and the preceding one.

Cor. 3. In a ſpheric triangle, the difference between the coverſed ſine of the vertical angle and the verſed ſine of the difference of the angles at the baſe is to the coverſed ſine of the baſe, as the difference between the coverſed ſine of the vertical angle and the verſed ſine of the ſum of the angles at the baſe is to the verſed ſine of the baſe.

This alſo follows, ex æquo, from the 1. Cor. of this Prop. and the corollary of the preceding.

LEM. XXVI.

HALF the ſum and half the difference between one magnitude and the exceſs of two other magnitudes are equal to the two exceſſes of half the ſum of the three magnitudes above each of the two.

LET AB, AC, CD be three magnitudes, and BC the exceſs of the two AB, AC; I ſay, half the ſum and half the difference of BC, CD are equal to the two exceſſes of half the ſum of the three magnitudes AB, AC, CD above each of the two AB, AC. *Fig.* 97.

BISECT BD in F, CD in G, and produce AD to E ſo that DE be equal to AB. Becauſe BD is double to FD, and CD double to GD, the remainder BC is double to the remainder FG. Wherefore FG being the half of BC, and CG the half of CD, the remainder CF is half the difference between BC, CD; and BF is half the ſum of the ſame. But becauſe AB is equal to DE, the whole AE is equal to the ſum of AB, AC, CD; and BF being alſo equal to FD, AF is equal to FE, viz. AF is the half of AE, or the half of the ſum of AB, AC, CD. Therefore BF, CF are alſo the exceſſes of half the ſum of AB, AC, CD above each of the two AB, AC.

LEM. XXVII.

HALF the difference between one magnitude and the ſum of two other magnitudes is equal to the difference between half the ſum of the three magnitudes and the one magnitude.

Fig. 98. LET AB, BC be two magnitudes, whoſe ſum is AC, and CD a third magnitude; I ſay, half the difference between AC, CD is equal to the difference between half the whole AD and CD.

MAKE DF (towards the parts of A) equal to AC, then FC is equal to the difference between AC, CD. Biſect AD in E, becauſe AE is equal to ED, and AC to DF, the remainder EC is equal to the remainder EF. Wherefore EC is the half of FC, and becauſe DE is the half of the whole AD, the ſame EC is the difference between the half of AD and CD.

PROP. XX. THEOREM.

IN a ſpheric triangle, the rectangle under the ſines of two of the ſides is to the ſquare of radius,

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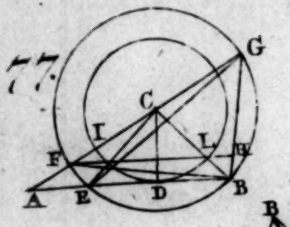

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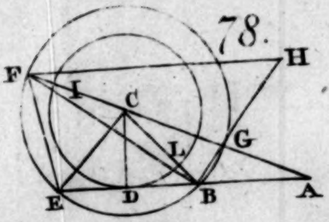

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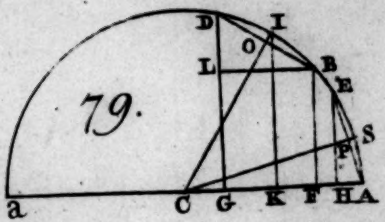

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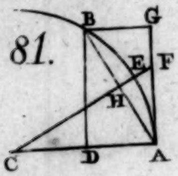

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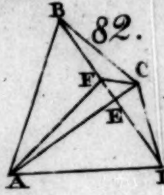

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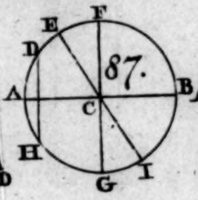

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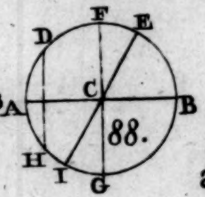

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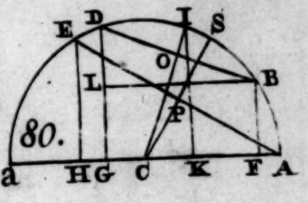

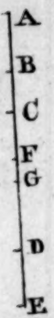

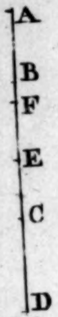

97. 98.

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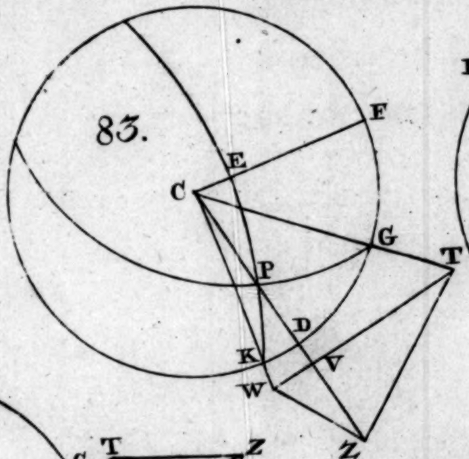

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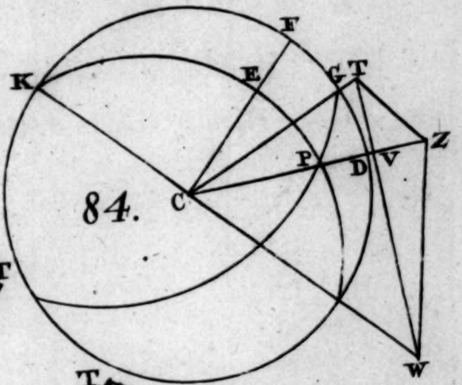

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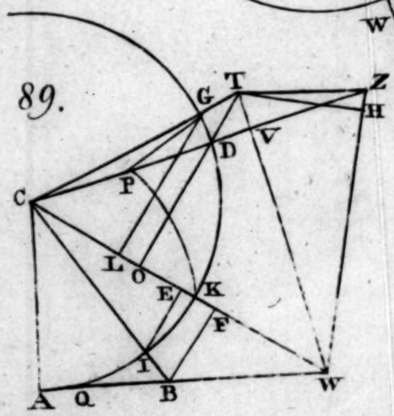

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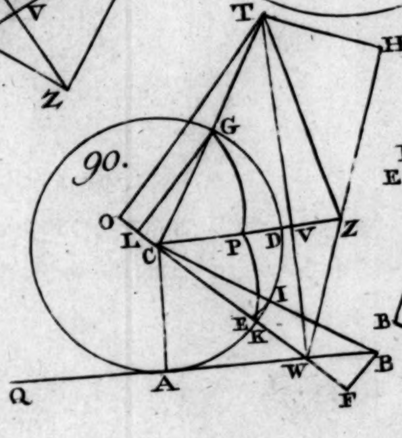

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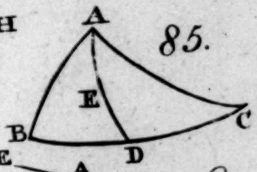

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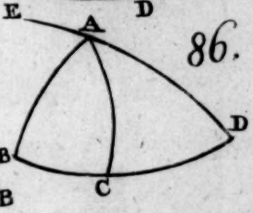


TAB. VIII. *pag.* 300.

dius, as the rectangle under the ſines of the two exceſſes of half the perimeter of the triangle above each of the ſides is to the ſquare of the ſine of half the angle included by the ſides.

Let AB, AC be two ſides of a ſpheric triangle ABC; as the rectangle under the ſines of AB, AC is to the ſquare of radius, ſo is the rectangle under the ſines of the two arches, whcih are the exceſſes of half the perimeter above each of the ſides AB, AC to the ſquare of the ſine of half the angle BAC. *Fig.* 41. 42.

As the rectangle under the ſines of AB, AC is to the ſquare of radius, ſo is the difference between the verſed ſine of BC and the verſed ſine of the difference of the ſides AB, AC to the verſed ſine of the angle BAC (Prop. XVIII.). But the difference between the verſed ſine of BC and the verſed ſine of the difference of the ſides AB, AC is to the verſed ſine of the angle BAC, (as the rectangle under the ſines of half the ſum and half the difference of the arches BC and the difference of the ſides is to the ſquare of the ſine of half the angle BAC (Lem. XXII.), viz.) as the rectangle under the ſines of the two exceſſes of half the perimeter of the triangle above each of the ſides AB, AC is to the ſquare of the ſine of half the angle BAC (Lem. XXVI.). Wherefore ex æquo, the rectangle under the ſines of AB, AC is to the ſquare of radius, as the rectangle under the ſines of the two exceſſes of half the perimeter above each of the ſides AB, AC is to the ſquare of the ſine of half the angle BAC.

PROP. XXI. THEOREM.

IN a ſpheric triangle, the rectangle under the ſines of two of the ſides is to the ſquare of radius, as the rectangle under the ſine of half the perimeter of the triangle, and the ſine of the exceſs of half the perimeter above the baſe is to the ſquare of the coſine of half the angle included by the ſides.

Fig. 41. 42. THE ſame things remaining, the rectangle under the ſines of AB, AC is to the ſquare of radius, as the rectangle under the ſines of half the perimeter of the triangle, and the exceſs of half the perimeter above the baſe BC is to the ſquare of the coſine of half the angle BAC.

FOR the rectangle under the ſines of AB, AC is to the ſquare of radius, (as the difference between the verſed ſines of BC and the ſum of the ſides AB, AC is to the coverſed ſine of the angle BAC (PROP. XIX.), viz.) as the rectangle under the ſines of half the ſum and half the difference of BC, and the ſides AB, AC taken together is to the ſquare of the coſine of half the angle BAC (1. Cor. Lem. XXII.). But half the ſum of BC and AB, AC taken together is half the perimeter of the triangle, and half the difference between BC and AB, AC taken together is equal to the exceſs of half the perimeter above BC (Lem. XXVII.). Wherefore the rectangle under the ſines of AB, AC is to the ſquare of radius, as the rectangle under the ſines of half the perimeter and the exceſs of half the perimeter above BC is to the ſquare of the coſine of half the angle BAC.

PROP. XXII. THEOREM.

IN a ſpheric triangle, the rectangle under the ſines of half the perimeter and the exceſs of half the perimeter above the baſe is to the rectangle under the ſines of the two exceſſes of half the perimeter above each of the ſides, as the ſquare of radius is to the ſquare of the tangent of half the vertical angle.

The rectangle under the ſines of half the perimeter and the exceſs of half the perimeter above the baſe is to the ſquare of the coſine of half the vertical angle, (as the rectangle under the ſines of the ſides is to the ſquare of radius (PROP. XXI.) ,viz.) as the rectangle under the ſines of the two exceſſes of half the perimeter above each of the ſides is to the ſquare of the ſine of half the vertical angle (PROP. XX.). Wherefore, by alternation, the rectangle under the ſines of half the perimeter and the exceſs of half the perimeter above the baſe is to the rectangle under the ſines of the two exceſſes of half the perimeter above each of the ſides, (as the ſquare of the coſine is to the ſquare of the ſine of half the vertical angle, viz.) as the ſquare of radius is to the ſquare of the tangent of half the vertical angle (LEM. XVI.).

PROP. XXIII. THEOREM.

IN a ſpheric triangle, the rectangle under the ſines of the angles at the baſe is to the ſquare of radius, as the rectangle under the coſines of the two exceſſes of half the ſum of the three angles above each of the angles at the baſe is to the ſquare of the coſine of half the baſe.

Fig. 41. 42. LET BC be the baſe of the ſpheric triangle ABC; I ſay, the rectangle under the ſines of the angles B, C is to the ſquare of radius, as the rectangle under the coſines of the two exceſſes of half the ſum of the three angles A, B, C above each of the angles B, C is to the ſquare of the coſine of half BC.

The rectangle under the ſines of the angles B, C is to the ſquare of radius, as the difference between the coverſed ſine of the angle A and the verſed ſine of the difference of the angles B, C is to the coverſed ſine of BC (Cor. Prop. XVIII.), viz.) as the rectangle under the coſines of half the ſum and half the difference of the angle A and the difference of the angles B, C is to the ſquare of the coſine of half BC (3. Cor. Lem. XXII.). But half the ſum and half the difference of the angle A, and the difference of the angles B, C are the ſame with the two exceſſes of half the ſum of the three angles above each of the angles B, C (Lem. XXVI.). Wherefore the rectangle under the ſines of the angles B, C is to the ſquare of radius, as the rectangle under the coſines of the two exceſſes of half the ſum of the three angles above each of the angles B, C is to the ſquare of the coſine of half BC. *Fig.* 41. 42.

PROP. XXIV. THEOREM.

IN a ſpheric triangle, the rectangle under the ſines of the angles at the baſe is to the ſquare of radius, as the rectangle under the coſines of half the ſum of the three angles and the exceſs of this half ſum above the vertical angle is to the ſquare of the ſine of half the baſe.

Fig. 41. 42. THE ſame things remaining, I ſay the rectangle under the ſines of the angles B, C is to the ſquare of radius, as the rectangle under the ſine of half the ſum of the three angles A, B, C and the ſine of the exceſs of this half ſum above the angle A is to the ſquare of the ſine of half the angle A.

FOR the rectangle under the ſines of the angles B, C is to the ſquare of radius, (as the difference between the coverſed ſine of the angle A and the verſed ſine of the ſum of the angles B, C is to the verſed ſine of BC (1. COR. PROP. XIX.), viz.) as the rectangle under the coſines of half the ſum and half the difference of the angle A, and the ſum of the angles B, C is to the ſquare of the ſine of half BC (2. COR. LEM. XXII.). But half the ſum of the angles A, and the ſum of the angles B, C is the ſame with half the ſum of the three angles A, B, C, and half the difference between the angle A, and the ſum of the two angles B, C is the ſame with the exceſs of half the ſum of the three angles above the angle A (LEM. XXVII.). Wherefore the rectangle under the ſines of the angles B, C is to the ſquare of radius, as the rectangle under the coſines of half the ſum of the three angles, and the exceſs of this half ſum above the angle A is to the ſquare of the ſine of half BC.

PROP. XXV. THEOREM.

IN a ſpheric triangle, the rectangle under the coſines of half the ſum of the three angles, and the exceſs of this half ſum above the vertical angle is to the rectangle under the ſines of the two exceſſes of this half ſum above each of the angles at the baſe, as the ſquare of radius is to the cotangent of half the baſe.

The rectangle under the ſines of half the ſum of the three angles and of the exceſs of this half ſum above the vertical angle is to the ſquare of the ſine of half the baſe, (as the rectangle under the ſine of the angles at the baſe is to the ſquare of radius (Prop. XXIV.), viz.) as the rectangle under the coſines of the two exceſſes of the half ſum above each of the angles B, C is to the ſquare of the coſine of half the baſe (Prop. XXIII.). Wherefore, by alternation, the rectangle under the coſines of half the ſum of the three angles, and the exceſs of this half ſum above the vertical angle is to the rectangle under the coſines of the two exceſſes of this half ſum above each of the angles at the baſe, as the ſquare of the ſine is to the ſquare of the coſine of half the baſe, viz. as the ſquare of radius is to the ſquare of the cotangent of half the baſe (1. Cor. Lem. XVI.).

Of

OF THE APPLICATION OF THE PRECEDING THEORY.

AS every ſpheric triangle conſiſts of ſix parts, if three of them be given, the triangle is determined. But the right angle in a right-angled triangle being a conſtant datum, it is required that only two of the remaining five be given, in order to determine the triangle.

IT has been uſual to divide Spheric Trigonometry into two parts, Right and Oblique-Angled; and to reſolve the firſt into ſixteen caſes, the latter into twelve. But this enumeration of caſes reſpects the *Quæſita,* while it is more natural to diſtinguiſh the caſe by the *Data,* and comprehend under each caſe the inveſtigation of all that is not given. On this principle each diviſion of Spheric Trigonometry reſolves itſelf into ſix caſes; viz.

IN RIGHT-ANGLED SPHERIC TRIANGLES.

CASE 1. When a ſide, and an adjacent angle are given.

Case 2. When a ſide, and an oppoſite angle are given.
Case 3. When a ſide, and the hypothenuſe are given.
Case 4. When the hypothenuſe, and an angle are given.
Case 5. When the two ſides are given.
Case 6. When the two angles are given.

In OBLIQUE-ANGLED SPHERIC TRIANGLES.

Case 1. When two ſides, and an oppoſite angle are given.
Case 2. When two angles, and an oppoſite ſide are given.
Case 3. When two ſides, and the included angle are given.
Case 4. When two angles, and the included ſide are given.
Case 5. When the three ſides are given.
Case 6. When the three angles are given.

In the following proportions, Co. denotes Complement, and *S.*, *T.*, *Coſ.*, *Cot.* denote Sine, Tangent, Coſine, Cotangent. The characters ſignifying proportion are in general uſe.

OF RIGHT-ANGLED SPHERIC TRIANGLES.

Fig. 99.

	Given.	Sought.	
1.	AB A	BC	AB Mid. Pt. Co. A, BC Extr. Conj. *Cot*. A: R:: *S*. AB: *T*. BC. BC is of the ſame affection with A (19. B. III.).
		AC	Co. A Mid. Pt. AB, Co. AC Extr. Conj. *T*. AB: R:: *Coſ*. A: *Cot*. AC. AC is leſs or greater than a quadrant, accordingly as A, C (21. B. III.), viz. as A and AB are of the ſame or different affection (19. B. III.).
		C	Co. C Mid. Pt. AB, Co. A Extr. Disj. R: *Coſ*. AB:: *S*. A: *S*. C. C is of the ſame affection with AB (19. B. III.).
2.	AB C	BC	BC Mid. Pt. AB, Co. C Extr. Conj. R: *T*. AB:: *Cot*. C: *S*. BC. BC may be either leſs or greater than a quadrant.

Given. Sought.

AC *S*. C : *S*. AB :: R : *S*. AC (2. B. VI.).

AC may be either leſs or greater than a quadrant.

A Co. C Mid. Pt.

AB, Co. A Extr. Diſj.

Coſ. AB : R : *Coſ*. C : *S*. A.

A may be either leſs or greater than a right angle.

3. AB BC Co. AC Mid. Pt.

AB, BC Extr. Diſj.

AC *Coſ*. AB : R :: *Coſ*. AC : *Coſ*. BC.

As AC is leſs or greater than a quadrant, BC will be of the ſame or different affection with AB (20. B. III.).

A Co. A Mid. Pt.

AB, Co. AC Extr. Conj.

R : *T*. AB :: *Cot*. AC : *Coſ*. A.

Becauſe the angles A, C are of the ſame or different affection, as AC is leſs or greater than a quadrant (21. B. III.), therefore alſo ſo will A and AB be (19. B. III.).

C *S*. AC : R :: *S*. AB : *S*. C (2. B. VI.).

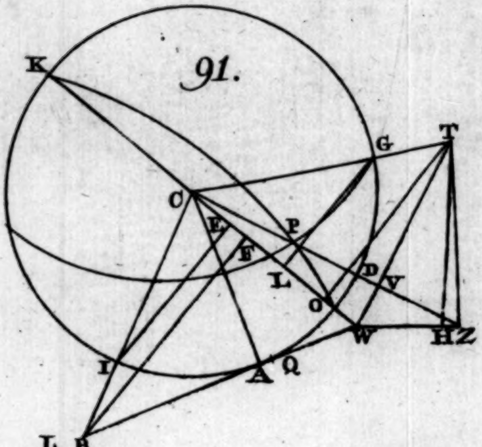
C is of the ſame affection with AB (19. B. III.).

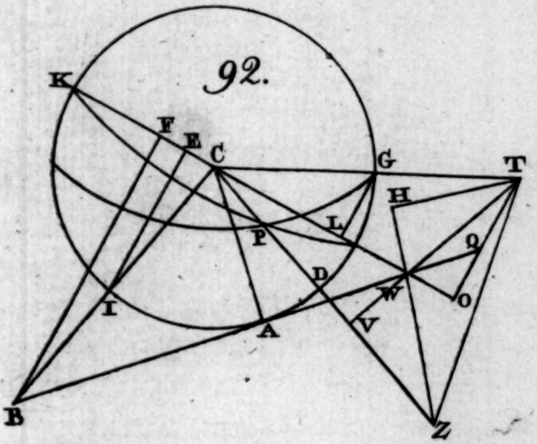
4. AC AB Co. A Mid. Pt.

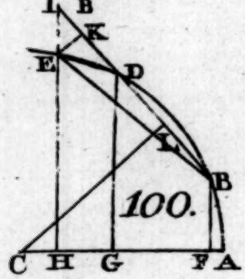
AB, Co. AC Extr. Conj.

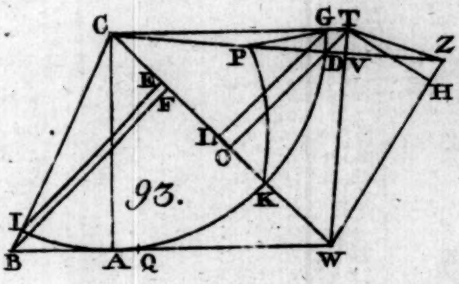
A *Cot*. AC : R :: *Coſ*. A : *T*. AB.

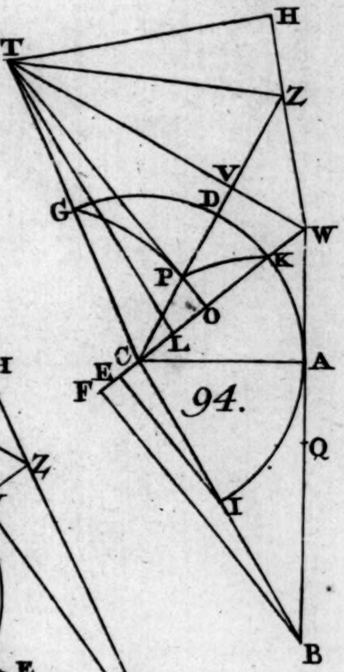
As AC is leſs or greater than a quadrant, the angles A, C (21. B. III.), and conſequently A and AB will be of the ſame or different affection (19. B. III.).

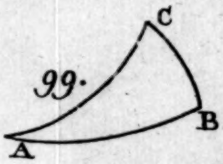


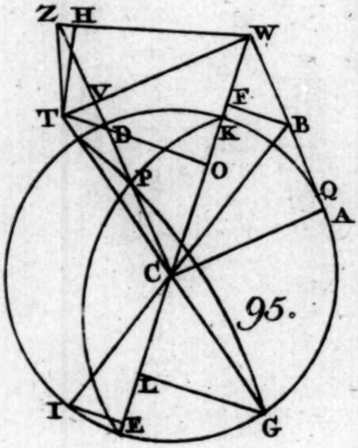


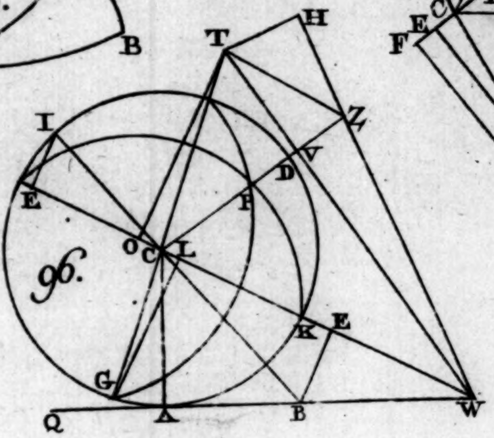












TAB. IX. *pag.* 312.

	Given.	Sought.	
		BC	R : *S*. AC :: *S*. A : *S*. BC (2. B. VI.). BC will be of the ſame affection with A (19. B. III.).
		C	Co. AC Mid. Pt. Co. A, Co. C Extr. Conj. *Cot*. A : R :: *Coſ*. AC : *Cot*. C. C will be of the ſame or different affection with A, accordingly as AC is leſs or greater than a quadrant (21. B. III.).
5.	AB	A	AB Mid. Pt. BC, Co. A Extr. Conj.
	BC		*T*. BC : R :: *S*. AB : *Cot*. A. A is of the ſame affection with BC (19. B. III.).
		AC	Co. AC Mid. Pt. AB, BC Extr. Disj. R : *Coſ*. AB :: *Coſ*. BC : *Coſ*. AC. AC is leſs or greater than a quadrant, as AB, BC are of the ſame or different affection (20. B. III.).
6.	A	AB	Co. C Mid. Pt. Co. A, AB Extr. Disj.
	C		*S*. A : R :: *Coſ*. C : *Coſ*. AB. AB is of the ſame affection with C (19. B. III.).

Given. Sought.

AC Co. AC Mid. Pt.

Co. A, Co. C Extr. Conj.

R: *Cot.* A:: *Cot.* C: *Coſ.* AC.

As A, C are of the ſame or different affection, AC will be leſs or greater than a quadrant (21. B. III.).

NOTE. Wherever the cotangent or coſine is found in any of the preceding proportions, they may be changed to the tangent and ſecant by the 1. and 2. COR. LEM. XVIII.

The ambiguity which is commonly aſcribed to the three parts of Caſe 2., means only that the Problem admits of two ſolutions, either of which anſwers the conditions of the data.

SOLUTION OF THE CASES OF OBLIQUE-ANGLED SPHERIC TRIANGLES.

Given.	Sought.		
1. AB AC B	C	*S*. AC : *S*. B :: *S*. AB : *S*. C (2. B. VI.). Half the angles B, C together will be of the ſame affection with reſpect to a quadrant, that half the ſides AB, AC together are (18. B. III.).	*Fig.* 41. 42.
	BC	*Cot*. AB : R :: *Coſ*. B : *T*. BD (3. B. VI.), and *Coſ*. AB : *Coſ*. AC : *Coſ*. BD : *Coſ*. CD (5. B. VI.). Then as the angles B, C are of the ſame or different affection, the perpendicular AD ſalls within or without the triangle (22. B. III.), and accordingly BC is the ſum or difference of BD, CD.	
	A	*Cot*. B : R :: *Coſ*. AB : *Cot*. BAD (3. B. VI.), and *Cot*. AB : *Cot*. AC :: *Coſ*. BAD : *Coſ*. CAD (6. B. VI.). Then as in the preceding, the ſum or difference of the angles BAD, CAD is the angle BAC. Or, BC being found, the angle A is had by the 2. B. VI. Or, the angle C being obtained, the angle A is directly found by the 14. or 15. B. VI.	

Given. Sought.

2. B, C, AB — AC — *S.* C : *S.* AB :: *S.* B : *S.* AC.

The half of AB, AC together are of the ſame affection as half the angles B, C together are (18. B. III.).

A — *Cot.* B : R :: *Coſ.* AB : *Cot.* BAD (3. B. VI.), and *Coſ.* B : *Coſ.* C :: *S.* BAD : *S.* CAD (10. B. VI.).

The ſum or difference of the angles BAD, CAD, accordingly as the angles B, C are of the ſame or different affection (22. B. III.), is the angle BAC.

BC — *Cot.* AB : R :: *Coſ.* B : *T.* BD (3. B. VI.), and *Cot.* B : *Cot.* C :: *S.* BD : *S.* CD (11. B. VI.).

When as before, the ſum or difference of BD, CD is the ſide BC.

Or, the angle A being found, BC is had by the 2. B. VI.

Or, AC being found, BC is directly obtained by the 16. or 17. B. VI.

3. AB, BC, B — A, C — $S.\ \frac{AB+BC}{2} : S.\ \frac{AB-BC}{2} :: Cot.\ \frac{B}{2} : T.\ \frac{A-C}{2}$ (14. B. VI.), and $Coſ.\ \frac{AB+BC}{2} : Coſ.\ \frac{AB-BC}{2} :: Cot.\ \frac{B}{2} : T.\ \frac{A+C}{2}$ (15. B. VI.).

Wherefore

Given. Sought.

Wherefore the ſum and difference of the angles A, C being given, the angles themſelves are given.

Or, *Cot.* AB : R :: *Coſ.* B : *T.* BD (3. B. VI.).

Then the difference between BD, BC is CD.

Again, *S.* BD : *S.* CD :: *Cot.* B : *Cot.* C (11. B. VI.).

As BD is leſs or greater than BC, the perpendicular will fall within or without the triangle, and therefore the angle C will be of the ſame or different affection with B (22. B. III.).

The angle C being found, the angle at A is obtained by the 2. B. VI.

AC BD, CD being found as in the preceding, *Coſ.* BD : *Coſ.* CD :: *Coſ.* AB : *Coſ.* AC.

As CD and AD, viz. as CD and B (19. B. III.) are of the ſame or different affection, AC will be leſs or greater than a quadrant (20. B. III.).

4. A, B, AB BC, AC, C $S.\,\frac{A+B}{2} : S.\,\frac{A-B}{2} :: T.\,\frac{AB}{2} : T.\,\frac{BC-AC}{2}$ (16. B. VI.), and $\text{Coſ.}\,\frac{A+B}{2} : \text{Coſ.}\,\frac{A-B}{2} :: T.\,\frac{AB}{2} : T.\,\frac{BC+AC}{2}$ (17. B. VI.). Whence the ſides BC, AC are given, and conſequently the angle C (2. B. VI.).

Or,

Given. Sought.

Or, *Cot.* B : R : : *Coſ.* AB : *Cot.* BAD (3. B. VI.), whence CAD is given. Then *Coſ.* BAD : *Coſ.* CAD : : *Cot.* AB : *Cot.* AC (6. B. VI.), and *S.* BAD : *S.* CAD : : *Coſ.* B : *Coſ.* C (10. B. VI.). Whence BC is obtained by the 2. B. VI.

As the angle BAD is leſs or greater than BAC, the perpendicular falls within or without the triangle, and accordingly the angle C is of the ſame or different affection with B (22. B. III.). Alſo AC will be leſs or greater than a quadrant, as the angles (CAD, ACD, that is, becauſe ACD is of the ſame affection with AD, and AD with the angle B) CAD, B are of the ſame or different affection.

5.	AB	A	Any one of the angles A, B, C is found either by the 20. 21. or 22. B. VI.
	AC	B	
	BC	C	
6.	A	AB	In the ſame manner any one of the ſides AB, AC, BC is found by the 23. 24. or 25. B. VI.
	B	AC	
	C	BC	

NOTE. The ambiguity which is aſcribed to Caſe 1., and part 1. and 3. of Caſe 2., means as in right-angled triangles no more, than that there are two ſolutions of the Problem, each of which anſwers the conditions required, viz. that the meaſure of the angle or the ſide ſought may be either leſs or greater than a quadrant.

THE END.

APPENDIX:

CONTAINING

THE SOLUTION OF A PROBLEM FOR ASCERTAINING THE APPARENT TIME OF AN OBSERVATION, TOGETHER WITH THE LATITUDE AND LONGITUDE OF THE OBSERVER.

LEMMA.

THE ratio of the exceſſes of the verſed ſines of three arches of a circle, together with the exceſſes of the arches themſelves, being given; the arches are required.

LET AB, AD, AE be three arches of a circle, whoſe ſines are BF, DG, EH perpendicular to the diameter AC, *Fig.* 100.

and

and verſed ſines AF, AG, AH. The ratio of FG to GH, together with the arches BD, DE in magnitude, are given in order to determine the arches themſelves.

JOIN BD, DE, BE, and from the centre C, as alſo the point E draw CL, EK perpendicular to BD, and let BD meet HE in I. The angles EBD, EDI are meaſured at the circumference by the arches ED, EB; and therefore at the centre by half theſe arches. Moreover, the angles CHI, CLI being right, the four points C, H, L, I are in a circle: wherefore the angle HIL is equal to the angle HCL, and conſequently is meaſured by the mean arch between AB, AD, or by half the ſum of the arches AB, AD. But KB, KD, KI being the cotangents of theſe angles to the radius EK, are proportional to the cotangents of half the arches ED, EB, and half the ſum of AB, AD, to any other radius. And, becauſe of the parallels, FG is to GH (as BD is to DI, viz. as BK — KD is to KD + IK, viz.) as the Cot. $\frac{1}{2}$ ED — Cot. $\frac{1}{2}$ EB is to the Cot. $\frac{1}{2}$ EB + Cot. $\frac{1}{2}\,\overline{AB + AD}$. But the ratio of FG to GH, as alſo the arches ED, EB being given; the mean arch between AB, AD, and conſequently each of the three arches AB, AD, AE, is given.

SCHOL. WITH little variation would the arches be given, en, if one of them were ſituated in the circumference towards oppoſite parts of A; but then what in the Lemma is termed an exceſs, would in this caſe become the ſum of two of the arches.

PROBLEM.

GIVEN three altitudes of an heavenly body, together with the times elapſed between the obſervations; to find the latitude of the obſerver, the declination of the heavenly body, and its diſtance from the meridian at the ſeveral times of obſervation.

It is required alſo that the theory of the Problem comprehend the change of the obſerver's latitude, and of the body's declination, during the intervals of the obſervation. But it is ſuppoſed, that the increments or decrements of latitude and declination in the two elapſed times are given; the former (in the change of place at ſea) from the reckoning of the ſhip, an error in which during ſo ſhort a run muſt be very inconſiderable, and the latter from an ephemeris.

Let the three altitudes be ſuppoſed to be each on the ſame ſide of the meridian, and their complements or the zenith diſtances to be denoted by A', A'', A''', of which A' is the greateſt, or that at the greateſt diſtance from the meridian; and let T', T'' denote the times (in parts of the body's diurnal motion) elapſed between the firſt and ſecond, the firſt and third altitudes. Theſe are given, and let H', H'', H''' repreſent the three correſponding diſtances of the heavenly object from the meridian; L', L'', L''', and D', D'', D''' the colatitudes and codeclinations, each increaſing from L', D'. Let a', a'', a''', and v', v'', v''' denote the verſed ſines of A', A'', A''', and of H', H'', H'''; alſo t', t'' the cotangents of $\frac{1}{2}T'$, $\frac{1}{2}T''$. Moreover, x, y denoting the verſed ſines of $L'' \mp D''$, let the differences between them and the verſed ſines of $L' \mp D'$, and of $L''' \mp D'''$ be ſignified by d', e', d'', e''; then will the verſed ſine of $L' - D' = x - d'$, of $L' + D' = y' + e'$, of $L''' - D''' = x + d''$, and of $L''' + D''' = y - e''$.

Because the zenith diſtance, co-latitude and co-declination of an heavenly body form a ſpheric triangle, one of whoſe angles, viz. oppoſite to the zenith diſtance, is the horary angle meaſured by the body's diſtance from the meridian; and in every ſpheric triangle, the difference between the verſed ſine of the baſe and verſed ſine of the difference of the ſides, is to the difference between the verſed ſine of the ſum of the ſides and verſed ſine of the difference of the ſides, as the verſed ſine of the vertical angle is to the diameter of the ſphere (by compoſition of the ratio in 2. Cor. XIX. B. VI.); the three ſubſequent equations, correſponding to the three obſervations, will be obtained, r being the radius of the ſphere.

$$\overline{a' - x + d'}\,.\ 2r = \overline{y - x + d' + e'}\,.\ v'$$

$$\overline{a'' - x}\,.\ 2r = \overline{y - x}\,.\ v''$$

$$\overline{a''' - x + d''}\,.\ 2r = \overline{y - x + d'' + e''}\,.\ v'''$$

Whence ſubſtracting the ſecond equation from the firſt, and the third from the ſecond, theſe two equations are obtained, viz.

$$\overline{a' - a'' + d'}\,.\ 2r = \overline{y - x}\,.\ \overline{v' - v''} + v'\,.\ \overline{d' + e'}$$

$$\overline{a'' - a''' + d''}\,.\ 2r = \overline{y - x}\,.\ \overline{v'' - v'''} + v'''\,.\ \overline{d'' + e''}$$

Or by tranſpoſition, and equal diviſion,

$$a' - a'' + d' - \frac{v'}{2r} \times \overline{d' + e'} = \frac{v' - v''}{2r} \times \overline{x - y}$$

$$a'' - a''' + d'' - \frac{v'''}{2r} \times \overline{d'' + e''} = \frac{v'' - v'''}{2r} \times \overline{x - y}$$

Whence becauſe equal things are proportional to equal things,

$$a''-a'''+d''-\frac{v'''}{2r}\times\overline{d''+e''} : a'-a''+d'-\frac{v'}{2r}\times\overline{d'+e'} ::$$

$(v''-v''' : v'-v'' ::)\ t'-t'' : t''+\text{Cot.}\ \frac{1}{2}\overline{\text{H}''+\text{H}'''}$ (Lem.)

Now ſuppoſing every thing in this proportion, except the Cot. $\frac{1}{2}\overline{\text{H}''+\text{H}'''}$, to be given; this cotangent will itſelf be given, and thence each of the diſtances from the meridian denoted by H', H'', H'''. But theſe being given, we have $y-x=\frac{2r}{v'-v''}\times\overline{a'-a''+d'-\frac{v'}{2r}\times\overline{d'+e'}}$, alſo $a''-x=v''\times\frac{y-x}{2r}=\frac{v''}{v'-v''}\times\overline{a'-a''+d'-\frac{v'}{2r}\times\overline{d'+e'}}$. Wherefore x and y are given.

But as in the preceding proportion ſome quantities are aſſumed as given, which do not immediately appear to be given, it will be neceſſary to ſhew on what principles this aſſumption is made. Becauſe the exceſs of the verſed ſines of two arcs will be nearly the ſame with the exceſs of the verſed ſines of two other arcs, which have the ſame difference, if they be not much greater or leſs than the former;

therefore if two arches, whoſe difference is given, be but inaccurately obtained, the exceſs of the verſed ſine of the one above the verſed ſine of the other may be aſſumed as a given quantity. If then L'', D'' be either aſſumed or obtained in any degree near to the truth, the correſponding values of L', D' and L''', D''' will be given, and therefore the difference between the arcs denoted by $L''\mp D''$, and and thoſe denoted by $L'\mp D'$, and $L'''\mp D'''$ being given, the exceſs of the verſed ſines of $L''\mp D''$ above thoſe of $L'\mp D'$ and $L'''\mp D'''$ are aſſumed as given quantities. Alſo, the approximate values of v', v''' being given or obtained, becauſe the terms of the proportion involving theſe quantities thus approximated to, are exceedingly ſmall in compariſon of the reſt, the whole proportion will be nearly given, and therefore a much nearer approach than in the firſt aſſumption to the true values of each of them will be acquired. Whence the proportion being corrected, all the deſiderata of the Problem would be determined to a degree of truth, beyond the limits of our preſent trigonometrical tables.

COR. IF the latitude and declination be ſuppoſed to be conſtant, then d', e', d'', e'' vaniſh, and in this caſe, $a''-a''' : a'-a'' :: t'-t'' : t''+\text{Cot. } \frac{1}{2}\overline{H''+H'''}$; alſo $y-x = \frac{2r}{v'-v''}\times\overline{a'-a''}$, and $a''-x = \frac{v''}{v'-v''}\times\overline{a'-a''}$.

Schol. Though it may be ſuppoſed that the latitudes, declinations and diſtances of the heavenly object from the meridian are given ſufficiently near for the firſt aſſumption; yet though every thing were utterly unknown except what enters into the immediate data of the Problem, theſe quantities would be found from the Corollary to a much greater degree of exactneſs than is required, for the firſt operation. The following example will illuſtrate both the Corollary and the Problem, every ſtep of which the reader can find no difficulty in referring to its proper authority.

But it is to be obſerved that inſtead of $\frac{v'}{2r}$, $\frac{v'''}{2r}$ are uſed $\frac{\overline{\text{S.}\,\frac{1}{2}\text{H}'}^{2}}{r^{2}}$, $\frac{\overline{\text{S.}\,\frac{1}{2}\text{H}'''}^{2}}{r^{2}}$ as being equivalent thereto; and further, that the tabular index in Sherwin is preſerved, the rule of which is, that a natural number anſwering to any ſine, &c. with four places of figures on the left hand of the comma, has 9 for its logarithmic index, and ſo it will be in proportion for any other number of places.

EXAMPLE.

FROM the nautical ephemeris of 1767, are taken three declinations of the ſun, adapted to three given periods of a given day, whence with three given latitudes are calculated the altitudes of the ſun correſponding to theſe times. The data of the Problem are therefore,

April 26th. 1767, at $\left.\begin{array}{l}22^{h}\,..\,0'\\ 22\,..\,30\\ 23\,..\,10\end{array}\right\}$ the verſed ſines of the zenith diſtances, or $\left.\begin{array}{l}a'\\ a''\\ a'''\end{array}\right\} = \left.\begin{array}{r}2988{,}589\\ 2649{,}407\\ 2344{,}358\end{array}\right\}$ Whence

$\left.\begin{array}{l}T'\\ T''\end{array}\right\} = \left.\begin{array}{l}0^{h}\,..\,30'\\ 1\,..\,10\end{array}\right\}$ or $\left.\begin{array}{l}7^{\circ}\,..\,30'\\ 17\,..\,30\end{array}\right\}$. Alſo $\left.\begin{array}{l}L'-L''\\ L''-L'''\end{array}\right\} = \left.\begin{array}{l}0^{\circ}\,..\,2'\\ 0\,..\,3\end{array}\right\}$

$\left.\begin{array}{l}D'-D''\\ D''-D'''\end{array}\right\} = \left.\begin{array}{l}0^{\circ}\,..\,0'\,..\,24''\\ 0\,..\,0\,..\,32\end{array}\right\}$ Therefore

$\left.\begin{array}{l}t' = (\text{Cot. } 3\,..\,45 =)\ 152570{,}52\\ t'' = (\text{Cot. } 8\,..\,45 =)\ 64971{,}043\end{array}\right\}$ Conſequently $t' - t''$ $= 87599{,}477$.

APPROXIMATION.

Ar. Comp^{t}. $a''-a'''$	305,049	1.5156304
$a'-a''$	339,182	8.5304328
$t'-t''$	87599,477	10.9425015
	97401,292	10.9885647
t'	64971,043	
Ct. $\frac{1}{2}\overline{H''+H'''}$	32430,249	$17^{\circ}\,..\,8'$
		5
H'''		$12\,..\,8$

v' -------- 1307,926 -- H' ---- 29 .. 38
v'' --------- 736,904 -- H'' ----- 22 .. 8

$v'-v''$ --------- 571,022 ------- 8.7566528
$r.\ \overline{a'-a''}$ ------------------------ 18.5304328

$\frac{1}{2}\overline{y-x}$ ---------5939,912 ------- 9.7737800
v'' ------------------------ 8.8674109

$a''-x$ ---------- 437,714 ------- 8.6411899
a'' -------------- 2649,407

x --------- 2211,693 ---- Vs. 38 .. 51
$y-x$ --------11879,824

y -------- 14091,517 ---- Vs. 114 .. 9

THEREFORE $L''=37° .. 39'$, $D''=76° .. 30'$. Consequently $L'=37° .. 41'$, $D'=76° .. 30' .. 24''$; and $L'''=37 .. 36$, $D'''=76° .. 29' .. 28$.

I. OPERATION.

d' ---- 2,919	d'' -- 4,504
e' ---- 6,369	e'' -- 9,381
$d'+e'$ ---- 9,288 -- 6.96792	$d''+e''$ - 13,885 -- 7.14255
$\overline{S.\frac{1}{2}H'^2}$ - (14..49) - 18.81555	$\overline{S.\frac{1}{2}H'''^2}$ - (6..4) - 18.04803
— ,607 --- 5.78347	— ,155 5.19058

d'

d' ----- $+2,919$ $d''+4,504$

$a'-a''+339,182$ $a''-a'''+305,049$

$\underline{}$ 341,494 309,398

Ar. Compt. -------------- 309,398 ------ 1.5094825

341,494 ------ 8.5333831

$t'-t''$ ---------------------- 10.9425015

96686,778 ---- 10.9853671

t' ---- 64971,043

Ct. $\frac{1}{2}\overline{H'+H'''}$ -- 31715,735 17 .. 30

5

H''' -- 12 .. 30

v' ---- 1339,746 - H' --- 30 .. 0

v'' ----- 761,205 - H'' --- 22 .. 30

578,541 ----- 8.7623341

$r.\ \overline{a'-a''}$ --------------- 18.5333831

$\frac{1}{2}\overline{y-x}$ 5902,677 ----- 9.7710490

v'' ------------------ 8.8815014

$a''-x$ --- 449,314 ----- 8,6525504

a'' - 2649,407

x -- 2200,093 ---- Vs. 38 .. 44 .. 25

$y-x$ - 11805,354

y - 14005,447 ---- Vs. 113 .. 36 .. 44

THEREFORE $L'' = 37 .. 26 .. 9$, $D'' = 76 .. 10 .. 34$.

BY this operation though the latitude and declination are found within 8 or 9 ſeconds, yet let it be ſuppoſed that $L'' = 37° .. 27'$, $D'' = 76 .. 11$, which is an error nearly of one whole minute. Therefore

$L' = 37° .. 29'$, $D' = 76 .. 11 .. 24$, and $L''' = 37°$ $24'$, $D''' = 76° .. 10' .. 28''$.

II. OPERATION.

d' ----	2,911		d'' --	4,491	
e' ----	6,395		e'' --	9,418	
$d'+e'$ ----	9,306	-- 6.96876	$d'+e'$ -	13,909	-- 7.14330
$\overline{\text{S.}\,\frac{1}{2}H'^2}$ -	(15..0)	- 18.82599	$\overline{\text{S.}\,\frac{1}{2}H'''^2}$	(6..15)	- 18.07379
	— ,623	5.79475		— ,165	-- 5.21709
d' ----	+ 2,911		d'' -	+ 4,491	
$a'-a''$ -	+ 339,182		$a''-a'''$ -	+ 305,049	
	341,470			309,375	

Ar. Comp^t. -------------	309,375 ------	1.5095148
	341,470 ------	8.5333525
$t'-t''$ -------------------		10.9425015
		96687,156

96687,156 ---- 10.9853688
t'' ------ 64971,043

Ct. $\frac{1}{2}\overline{H''+H'''}$ -- 31716,113 ------ 17° .. 30′
5 .. 0
H''' - 12 .. 30
v' ----- 1339,746 - H' -- 30 .. 0
v'' ----- 761,205 - H'' -- 22 .. 30
578,541 ----- 8.7623341
$r.\,\overline{a'-a''}$ ---------------- 18.5333525
$\frac{1}{2}\overline{y-x}$ -- 5902,261 ----- 9.7710184
v'' ---------------- 8.8815014
$a''-x$ --- 449,283 ----- 8.6525198
a'' -- 2649,407
x -- 2200,124 --- Vs. 38 .. 44 .. 26
$y-x$ - 11804,522
y - 14004,646 --- Vs. 113 .. 36 .. 26

THEREFORE $L''=37° .. 26'$, $D''=76° .. 10' .. 26''$, which are true to a ſingle ſecond. For the aſſumed latitudes were

52 .. 32	at	22^h .. 0	Whence from the ephemeris the declinations were found to be
52 .. 34		22 .. 30	
52 .. 37		23 .. 10	

13 .. 49 .. 10
13 .. 49 .. 34
13 .. 50 .. 6.

SCHOL. THE declination of the ſun being therefore found true to a ſingle ſecond, if from an ephemeris the apparent time, or the diſtance of the ſun from the meridian of any other place be obtained, when the declination is the ſame with that found by the Problem, the ſum or difference of this time and the apparent time of the obſerver's place found by the Problem, will be the true difference of longitude of the two places.

SOME difference will obtain in the Problem, accordingly as the latitude and declination are both increaſing or both decreaſing, or the one increaſing and the other decreaſing; or as the altitudes are all on the ſame ſide of the meridian, or on different ſides: but this difference might be eaſily aſcertained, by a table adjuſted to all the variety which can take place.

REMARKS.

THE Author does not flatter himſelf that any great advantage will be derived to the public from the preceding Problem; but to the curious it may not be unacceptable, and with this view a ſpecimen of its ſolution is annexed, as a proof of what the Problem, though founded on the principle of approximation, is capable of performing. For it is certain that if the data of the Problem could be given to the accuracy of a ſingle ſecond, the latitude, longitude and apparent time of the obſerver would be obtained with the ſame accuracy; and it has this advantage above every other method which I have ſeen, that from the ſame ſimple data, are at once obtained theſe three great deſiderata. *Francis à Schooten* and Sir Iſaac Newton have given ſolutions of the ſame Problem, but by a very different analyſis, and only on the ſuppoſition that the latitude and declination ſuffer no change during the time of the obſervations. No author, that I know of, has taken into conſideration, the quantity of variation in the latitude and declination; though this be abſolutely eſſential to a perfect ſolution. Yet it is freely acknowledged that if the ſun be the ſubject of obſervation, the whole is but a ſpeculative nicety; for it is not to be expected that even at land, and with the beſt inſtruments, the altitude of the ſun can be obſerved with that accuracy, as to determine the longitude of the obſerver from this Problem. Becauſe the whole baſis for 360 degrees of longitude is merely the change of the ſun's declination in twenty-four hours, the medium of which being leſs than twenty minutes, an error of four ſeconds in the declination would be productive of an error of more than one degree of longitude. But in the moon's motion, whoſe declination varies ſometimes more than ſix degrees in twenty-four hours, it would require an error of one minute to produce an error of a degree in longitude; and it may be preſumed that the accuracy of obſervations at land is more than ſufficient to avoid ſo great an error as this, nay it is probable that not one fourth part of this error would need to be apprehended. By this method therefore the longitude of places aſhore might be determined at all times with conſiderable accuracy. But if the latitude and meridian of the place had been determined, all the trouble of the preceding calculation would be avoided, by finding the meridian altitude of the moon. For from thence the declination of the moon at the time of her croſſing the meridian would immediately be given; and from the ephemeris of a place, the diſtance of the moon from that place's meridian, when her declination is the ſame with the one found, being obtained; this diſtance would be the difference of longitude between the two places.

THESE data however cannot be obtained at ſea, as from the imperfect definition of the horizon, the altitude of the moon cannot be taken with any certainty by Hadley's quadrant. But it appears to the author to be no inſuperable difficulty to conſtruct an inſtrument which ſhould give the altitude of the moon at any time with as much certainty as that of the ſun. If this could be accompliſhed, the Problem here given to the public might be found not unworthy its attention. For it is not ſubject to ſome conſiderable diſadvantages, which attend the preſent lunar method of determining a ſhip's place at ſea. The latitude, the apparent time, and the declination depend for their truth upon the ſame data, and are each affected only by the error which enters into the ſame obſervations. But the preſent method has the ſame imperfection to anſwer for in the obſervation of the moon's diſtance from a ſtar; and the apparent time, on which the concluſion eſſentially depends, is affected with an uncertainty proper to itſelf, and which appears to me to be ſuch as might be expected to vitiate the whole proceeding. I have long thought this an obſtacle of the greateſt conſideration to the lunar method for diſcovering the longitude at ſea, which is at preſent ſo warmly purſued. I may be miſtaken, and ſhould be glad to find myſelf ſo, as I have no view but to truth and utility. But if otherwiſe, the public attention has been too long employed in an impracticable purſuit, and diverted from inveſtigating this important Problem in other ways, if indeed in any way the ſolution of it be poſſible.

THE END.

BRITISH 24 DE 75 MUSEUM

ERRATA.

Page	Line *from the top.*	
22	2	*For* ALI, AIL, *r.* AIL, ALI.
—	23 & 24	FHL, DHI, *r.* FHI, DHL.
24	15	E, *r.* P.
32	11	*Shift the comma following* centre *to follow* alſo
51	5	COU, *r.* CUH.
—	17	CHU, *r.* CUH.
76	4	FF, *r.* EF.
130	15	*Between* TO, Oo *inſert* is to
141	9	Ta, *r.* TL.
148	9	projection, *r.* primitive
253	9	OCG, *r.* OCA
256	23	*Dele* together
277		*In the* Cor. *for* baſe *r.* vertical angle, *and for* ſum or difference of the ſegments, *r.* ſum or difference of the angles at the vertex.
282	2 & 5	AD, BD, *r.* BD, CD.
329	10	$0''\ 30'$ *r.* $0^h\ 30'$.